OXFORD IB DIPLOMA PROGRAMME

ADDITIONAL EXERCISES

MATHEMATICS:
ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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1.1 Sequences, series and sigma notation

1 For each of the sequences below, write the next three terms and find the general term.

a $3, 9, 15, \ldots$

b $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots$

c $-1, -3, -9, \ldots$

2 For each general term below, write the first three terms of the sequence.

a $\{u_n\} = \{-3n + 2\}, n \in \mathbb{Z}^+$

b $\{u_n\} = \{(-1)^{n-1}2^n\}, n \in \mathbb{Z}^+$

3 For each series written in sigma notation below, write the first 4 terms of the sequence.

a $\sum_{i=2}^{8} 3i - 1$

b $\sum_{i=1}^{20} 2^i + 3$

c $\sum_{i=1}^{5} (-1)^i i^3$

d $\sum_{i=3}^{10} \frac{i-1}{i+1}$

4 Write each of the following series in sigma notation.

a $5 + 13 + 21 + 29$

b $-3 + 6 - 12 + 24 - 48$

c $4 + 8 + 12 + \cdots$

Answers

1 **a** 21, 27, 33

$3, 9, 15, \ldots$

$3x1, 3x3, 3x5, \ldots$

Since the sequence is 3 times the positive odd integers:

$u_n = 3(2n-1), n \in \mathbb{Z}^+$

b $\frac{7}{4}, \frac{9}{4}, \frac{11}{4}$

$\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots$

$u_n = \frac{2n-1}{4}, n \in \mathbb{Z}^+$

c -27, -81, -243

$-1, -3, -9, \ldots$

$-1 \times 3^0, -1 \times 3^1, -1 \times 3^2$

$u_n = (-1) \times 3^{n-1}, n \in \mathbb{Z}^+$

2 **a** $u_1 = -3(1) + 2 = -1$

$u_2 = -3(2) + 2 = -4$

$u_3 = -3(3) + 2 = -7$

b $u_1 = (-1)^{1-1}2^1 = 2$

$u_2 = (-1)^{2-1}2^2 = -4$

$u_3 = (-1)^{3-1}2^3 = 8$

3 **a** $3(2) - 1,\ 3(3) - 1,\ 3(4) - 1,\ 3(5) - 1$

5,8,11,14

b $2^1 + 3,\ 2^2 + 3, 2^3 + 3,\ 2^4 + 3$

5, 7, 11, 19

c $(-1)^1(1)^3,\ (-1)^2(2)^3,\ (-1)^3(3)^3,\ (-1)^4(4)^3$

-1, 8, -27, 64

d $\frac{3-1}{3+1}, \frac{4-1}{4+1}, \frac{5-1}{5+1}, \frac{6-1}{6+1}$

$\frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}$

$\frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}$

4 **a** $5 + 13 + 21 + 29$

(5+0) + (5+8) + (5+16) + (5+24)

(5+4x0) + (5+4x2) + (5+4x4) + (5+4x6)

$\sum_{n=1}^{4} 5 + 4(2n - 2) = \sum_{n=1}^{4} 5 + 8n - 8 = \sum_{n=1}^{4} 8n - 3$

b $-3 + 6 - 12 + 24 - 48$

$(-1)^1(3 \times 1), (-1)^2(3 \times 2), (-1)^3(3 \times 4), (-1)^4(3 \times 8), (-1)^5(3 \times 16)$

$(-1)^1(3 \times 2^0), (-1)^2(3 \times 2^1), (-1)^3(3 \times 2^2), (-1)^4(3 \times 2^3), (-1)^5(3 \times 2^4)$

$\sum_{n=1}^{5}(-1)^n\,(3 \times 2^{n-1})$

c $4 + 8 + 12 + \cdots$

$(4 \times 1) + (4 \times 2) + (4 \times 3) + \cdots$

$$\sum_{n=1}^{\infty} 4n$$

1.2 Arithmetic and geometric sequences and series

1 For each arithmetic progression below, find the value of the common difference and the general term then determine how many terms are in each progression.

a $3, 7, 11, \ldots, 75$

b $-12, -14, -16, \ldots - 40$

c $0.10, -2.4, -4.9 \ldots - 64.9$

2 Give that an arithmetic progression has a third term of 4 and a sixth term of 184, find the value of the 12$^{\text{th}}$ term.

3 The Summer Olympics are held every four years. They were held in Rio De Janeiro in 2016. When will they be held for the first time after 2060?

4 The sum of three consecutive terms of an arithmetic sequence is 78 and their product is 17 472. Find the three numbers.

5 A geometric sequence has a common ratio of $-\frac{3}{4}$ and the sixth term is -1280. Find:

a The first term.

b The 20$^{\text{th}}$ term.

c The sum of the first 8 terms.

6 In a geometric sequence, $u_1 = 5$ and $u_3 = 1280$. The last term of the sequence is 20 480. Find the number of terms in the sequence.

7 Given a geometric sequence whose first term is -2 and has common ratio of $\frac{1}{4}$,find which term has a value of $\frac{-1}{8192}$.

8 For each series below, suggest whether it is arithmetic, geometric or neither, and find the sum indicated.

a $\frac{1}{15} + \frac{8}{15} + \frac{13}{15} + \cdots, S_6$

b $12 + 18 + 24 + \cdots + 300$

c $\sum_{k=1}^{6} (-1)^k (2)^{k+1}$

9 Insert four terms between 81 and $\frac{1}{729}$ to form a geometric sequence.

10 How many multiples of 3 are there less than 1000?

11 Evaluate $\sum_{n=1}^{15} -3(2)^n$.

12 Decide if the following geometric sequences are converging. If so, find the sum.

a $0.25 + 0.375 + 0.5625 + \cdots$

b $-\frac{3}{8} + \frac{9}{32} - \frac{27}{128} + \cdots$

c $\sum_{n=1}^{\infty} -4\left(\frac{1}{2}\right)^n$

d $(9x - 9) + (3x - 3) + (x - 1) + \cdots$

e $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{2}}+\frac{4}{\sqrt{2}}+\cdots$

f $\frac{1}{\sqrt{2}}-\frac{1}{2\sqrt{2}}+\frac{1}{4\sqrt{2}}-\cdots$

13 A ball is dropped from a height of 1.75 meters. With each bounce, the ball rebounds 70% of its height.

a Sketch a diagram of the first 3 bounces. Include the heights of each bounce.

b How many times does the ball need to hit the ground before its bounce is less than 20 centimetres?

c What is the total distance the balls travels until it comes to a stop?

14 A car deprecates at a rate of 21.5% per year. If you bought a new car today for $35 690, how much will it be worth after six years?

15 Michael checks the balance of his savings account and sees that he has $1800. If the interest rate is 1.25% per annum, compounded monthly, how did Michael originally deposit four years ago if the has not withdrawn or deposited any money since then?

16 If you invest 4500 Swiss Francs today in an investment that pays 2.25% per annum, compounded quarterly, how long will it take for your investment to double in value?

17 The first term of a geometric sequence is 32 and the common ratio is 0.5. Find the largest term that is smaller than $\frac{1}{1000}$.

18 A geometric series has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the common ration and hence, find the first term of the sequence.

19 The sum of an infinite geometric series is 81 with a common ratio of $\frac{1}{4}$. Find the first three terms of this series.

Answers

1 **a** $d = 7 - 3 = 4$

$u_n = 3 + (n-1)4 = 3 + 4n - 4 = 4n - 1$

$75 = 4n - 1$

$76 = 4n$

$n = 19$

b $d = -14 - -12 = -2$

$u_n = -12 + (n-1)(-2) = -12 - 2n + 2 = -10 - 2n$

$-40 = -10 - 2n$

$-30 = -2n$

$n = 15$

c $d = -2.4 - 0.1 = -2.5$

$u_n = 0.10 + (n-1)(-2.5)$

$u_n = 0.10 - 2.5n + 2.5$

$u_n = -2.5n + 2.6$

$-64.9 = -2.5n + 2.6$

$67.5 = -2.5n$

$n = 27$

2 $u_3 = 4, u_6 = 184$

$184 = 4 + (4-1)d$

$184 = 4 + 3d$

$180 = 3d$

$d = 60$

$u_{12} = 4 + (10-1)(60)$

$u_{12} = 544$

3 $2060 = 2016 + (n-1)(4)$

$44 = 4(n-1)$

$11 = n - 1$

$n = 12$

They will next be held in 2064.

4. Let the three terms be $x - d, x, x + d$.

Using the sum:

$x - d + x + x + d = 78$

$3x = 78$

$x = 26$

This makes the three terms: $26 - d, 26, 26 + d$.

Using the product:

$26(26 - d)(26 + d) = 17472$

$26(26^2 - d^2) = 17472$

$676 - d^2 = 672$

$d^2 = 4$

$d = 2$

The three terms are 24, 26, 28.

5 **a** $u_n = u_1 r^{n-1}$

$-1280 = u_1\left(-\frac{3}{4}\right)^{6-1}$

$-1280 = u_1\left(-\frac{3}{4}\right)^{5}$

$-1280 = u_1\left(\frac{-243}{1024}\right)$

$u_1 = \frac{1310720}{243}$

b $u_{20} = \left(\frac{1310720}{243}\right)\left(-\frac{3}{4}\right)^{20-1}$

$u_{20} \approx -22.8$

c $S_n = \frac{u_1(1-r^n)}{1-r}$

$S_8 = \frac{\frac{1310720}{243}\left(1-\left(-\frac{3}{4}\right)^8\right)}{1-\left(-\frac{3}{4}\right)}$

$S_8 \approx 2773.7$

6 $1280 = 5r^{3-1}$

$256 = r^2$

$r = 16$

$20480 = 5(16)^{n-1}$

$4096 = (16)^{n-1}$

$16^3 = (16)^{n-1}$

$3 = n - 1$

$n = 4$

7 $\frac{-1}{8192} = -2\left(\frac{1}{4}\right)^{n-1}$

$\frac{1}{16384} = \left(\frac{1}{4}\right)^{n-1}$

$\left(\frac{1}{4}\right)^7 = \left(\frac{1}{4}\right)^{n-1}$

$7 = n - 1$

$n = 8$

8 **a** $\frac{8}{15} - \frac{1}{15} \neq \frac{13}{15} - \frac{8}{15}, \therefore$ not arithmetic

$\frac{\frac{8}{15}}{\frac{1}{15}} \neq \frac{\frac{13}{15}}{\frac{8}{15}}, \therefore$ not geometric

b $18 - 12 = 24 - 18, \therefore$ arithmetic

To find n:

$u_n = u_1 + (n-1)d$

$300 = 12 + (n-1)6$

$288 = 6(n-1)$

$48 = n - 1$

$n = 49$

$S_{49} = \frac{49}{2}(12 + 300)$

$S_{49} = \frac{49}{2}(312)$

$S_{49} = 7644$

c Geometric with $r = \frac{(-1)^2(2)^{2+1}}{(-1)^1(2)^{1+1}} = \frac{8}{-4} = -2$ and $u_1 = (-1)^1 2^{1+1} = -4$.

$S_6 = \frac{-4(1-(-2)^6)}{1--2}$

$S_6 = \frac{-4(1-64)}{3}$

$S_6 = \frac{-4(-63)}{3}$

$S_6 = \frac{252}{3}$

$S_6 = 84$

9 $\frac{1}{729} = 81(r)^{6-1}$

$\frac{1}{59049} = r^5$

$\left(\frac{1}{9}\right)^5 = r^5$

$r = \frac{1}{9}$

$u_2 = 81\left(\frac{1}{9}\right) = 9$

$u_3 = 9\left(\frac{1}{9}\right) = 1$

$u_4 = 1\left(\frac{1}{9}\right) = \frac{1}{9}$

$u_5 = \frac{1}{9}\left(\frac{1}{9}\right) = \frac{1}{81}$

10 The multiples of 3 s are 3, 6, 9, 12, …

$3 + (n-1)3 > 1000$

$3(n-1) > 997$

$n - 1 > 332.3333 \ldots$

$n > 333.3333 \ldots$

There are 333 terms.

11 $u_1 = -3(2)^1 = -6$

$u_2 = -3(2)^2 = -12$

$r = -\frac{12}{-6} = 2$

$S_{15} = \frac{-6(1-(2)^{15})}{1-2}$

$S_{15} = \frac{-6(1-32768)}{-1}$

$S_{15} = 6(32767)$

$S_{15} = 196\,602$

12 a $r = \frac{0.375}{0.25} = 1.5, \therefore$ The series is not converging.

b $r = \frac{\frac{9}{32}}{\frac{-3}{8}} = \frac{9}{32} \times \frac{8}{-3} = \frac{-3}{4}$

$S_\infty = \frac{-\frac{3}{8}}{1 - -\frac{3}{4}}$

$S_\infty = \frac{-\frac{3}{8}}{\frac{7}{4}}$

$S_\infty = -\frac{3}{8} \times \frac{4}{7}$

$S_\infty = -\frac{3}{14}$

c $r = \frac{-4\left(\frac{1}{2}\right)^2}{-4\left(\frac{1}{2}\right)^1} = \frac{-1}{-2} = \frac{1}{2}$

$S_\infty = \frac{-2}{1-\frac{1}{2}}$

$S_\infty = \frac{-2}{\frac{1}{2}}$

$S_\infty = -4$

d $r = \frac{3x-3}{9x-9} = \frac{3(x-1)}{9(x-1)} = \frac{1}{3}$

$S_\infty = \frac{9x-9}{1-\frac{1}{3}}$

$S_\infty = \frac{9x-9}{\frac{2}{3}}$

$S_\infty = \frac{3}{2}(9x-9)$

$S_\infty = \frac{27}{2}x - \frac{27}{2}$

e $r = \frac{\frac{2}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 2, \therefore$ The series is not converging.

f $r = \frac{\frac{-1}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -\frac{1}{2}$

$S_\infty = \frac{\frac{1}{\sqrt{2}}}{1 - -\frac{1}{2}}$

$S_\infty = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{2}}$

$S_\infty = \frac{\frac{1}{\sqrt{2}}}{\frac{3}{2}}$

$S_\infty = \frac{1}{\sqrt{2}} \times \frac{2}{3}$

$S_\infty = \frac{2}{3\sqrt{2}}\left(= \frac{\sqrt{2}}{3}\right)$

13 a Diagram of a bouncing ball with at least three bounces. The bounces should be labelled as follows: 1.75m, 1.225m, 0.8575m.

b $u_1 r^{n-1} < 0.2$

$1.75(0.7)^{n-1} < 0.2$

$(0.7)^{n-1} < 0.114285 \ldots$

By GDC, $n > 7$

c Total distance $= 2S_\infty - u_1$

Total distance $= 2\left(\frac{1.75}{1-0.7}\right) - 1.75$

Total distance $= \frac{35}{3} - \frac{7}{4}$

Total distance $= \frac{119}{12} \approx 9.92m$

14 $u_7 = 35690(1 - 0.215)^{7-1}$

$u_7 = 35690(0.785)^6$

$u_7 \approx \$8351.50$

15 $1800 = u_1\left(1 + \frac{0.0125}{12}\right)^{12\times 4}$

$1800 = u_1\left(1 + \frac{0.0125}{12}\right)^{48}$

$1800 = u_1(1.0512437 \ldots)$

$u_1 \approx \$1712.26$

16 $9000 = 4500\left(1 + \frac{0.0225}{4}\right)^{4n}$

$2 = \left(1 + \frac{0.0225}{4}\right)^{4n}$

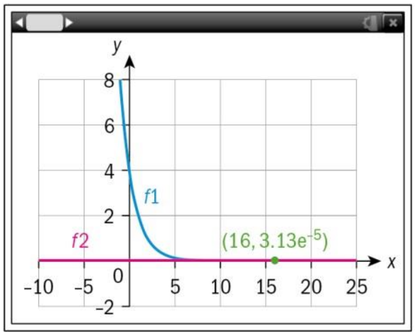
$2 = (1.005625)^{4n}$

By GDC, 30.9 years

17 $32(0.5)^{n-1} < 0.001$

$(0.5)^{n-1} < 0.00003125$

By GDC, $n = 16$



Note that the term with $n = 16$ is $32(0.5)^{15} = \frac{1}{1024}$.

18 Let x be the first term.

$27 = \frac{x}{1-r}$

$27(1-r) = x$

$x + x \cdot r = 15$

$27(1-r) + (27(1-r)) \cdot r = 15$

$27 - 27r + 27r - 27r^2 = 15$

$27 - 27r^2 = 15$

$-27r^2 = -12$

$r^2 = \frac{12}{27} = \frac{4}{9}$

$r = \pm\frac{2}{3}$

Since the terms must all be positive, $r = \frac{2}{3}$.

$x = 27\left(1 - \frac{2}{3}\right)$

$x = 27\left(\frac{1}{3}\right)$

$x = 9$

19 $81 = \frac{u_1}{1-\frac{1}{4}}$

$81 = \frac{u_1}{\frac{3}{4}}$

$u_1 = \frac{243}{4}$

$u_2 = u_1\left(\frac{1}{4}\right)^{2-1} = \frac{243}{16}$

$u_3 = u_1\left(\frac{1}{4}\right)^{3-1} = \frac{243}{64}$

1.3 Proof

1 Prove that the square of any even number is also even.

2 Prove that if n is an odd integer then n^2 is also an odd integer

3 Let $x, y,$ and z be integers. Prove directly that if x divides y *and* x divides z, then x also divides $y + z$.

4 Let m and n be perfect square integers. Show that mn is also a perfect square integer.

5 If n is an even integer, prove that $7n + 4$ is also an even integer.

6 Show that if n is any even integer, then $(-1)^n = 1$.

7 Prove that for all integers n, $4(n^2 + n + 1) - 3n^2$ is a perfect square.

8 Prove that the product of any two rational numbers is a rational number.

9 Prove that if $a, b,$ and c are odd integers, then the equation $ax^2 + bx + c = 0$ has no integer solution.

10 Prove that the statement $3(x-2) - 4(3x+5) + 2x - 2 = -7(x+4)$ is true for all values of x.

11 a Prove that $\frac{x-2}{x} \div \frac{3x-6}{x^2+x} = \frac{x+1}{3}$

b For what values of x does this mathematical statement **not** hold true?

12 Use contraction to show that there exists no integers n and m for which $18n + 6m = 1$.

13 Prove that if $a, b, \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

14 Find a counterexample that disproves the statement "For all $n \in \mathbb{R}$, $n^2 + 4 > 5$.

15 Prove by induction: $n! > 3^n$ for $n \geq 7, n \in \mathbb{Z}$.

16 Prove by induction: $(2n)! \geq 2^n (n!)^2, n \in \mathbb{Z}^+$.

17 Prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z}^+$.

18 Use mathematical induction to prove that $n^3 + 11n$ is divisible by 3 for $n \in \mathbb{Z}^+$.

19 Prove by induction that $n(n^2 + 5)$ is divisible by 6 for $n \in \mathbb{Z}^+$.

20 Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, n \in \mathbb{Z}^+$.

21 Prove that $\sum_{i=1}^{n} \frac{1}{(3i-1)(3i+2)} = \frac{n}{6n+4}, n \in \mathbb{Z}^+$.

22 A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for all $n \in \mathbb{Z}^+$. Prove that

$u_n = 2^n - 1$.

23 Prove by induction that the formula for the sum of a finite geometric series holds true for all $n \in \mathbb{Z}^+$. (IB Q)

Answers

1 Assume n is an even number, $\therefore n = 2k$, for some integer k. Then

$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since this is divisible by 2, n^2 is even.

2 Assume n is an odd integer, $\therefore n = 2k + 1$, for some integer k. Then

$n^2 = (2k + 1)^2$

$n^2 = (2k + 1)(2k + 1)$

$n^2 = 4k^2 + 4k + 1$

$n^2 = 2(2k^2 + 2k) + 1$

$\therefore n^2$ is an odd integer.

3 Since x divides y, then $y = ax$, for some integer a.

Also, since x divides z, then $z = bx$, for some integer b.

$y + z = ax + bx = x(a + b)$

Hence x divides $y + z$ since $a + b$ is an integer.

4 $m = k^2$, for some integer k and $n = l^2$, for some integer l.

$mn = k^2l^2 = (kl)^2$

Since kl is an integer, mn is a perfect square.

5 Assume n is an even number, $\therefore n = 2k$, for some integer k.

Then, $7n + 4 = 7(2k) + 4$

$7n + 4 = 14k + 4$

$7n + 4 = 2(7k + 2)$

Since k is an integer, so is $7k + 2$. Since $7n + 4$ is divisible 2, it is an even integer.

6 Assume n is an even number, $\therefore n = 2k$, for some integer k.

$(-1)^n = (-1)^{2k}$

$(-1)^n = ((-1)^2)^k$

$(-1)^n = (1)^k$

$(-1)^n = 1$

QED

7 $4(n^2 + n + 1) - 3n^2 = 4n^2 + 4n + 4 - 3n^2$

$4(n^2 + n + 1) - 3n^2 = n^2 + 4n + 4$

$4(n^2 + n + 1) - 3n^2 = (n + 2)(n + 2)$

$4(n^2 + n + 1) - 3n^2 = (n + 2)^2$

Since n is an integer, $n+2$ is also an integer and thus $4(n^2+n+1)-3n^2$ is a perfect square.

8 Assume a and b are rational numbers.

$a=\frac{c}{d}$ and $b=\frac{e}{f}, d.f\neq 0$

$ab=\left(\frac{c}{d}\right)\left(\frac{e}{f}\right)=\frac{ce}{df}$

Since ce and df are both integers as they are the product of integers, and $df\neq 0$ as it is the product of two non-zero integers, ab is a quotient of rational numbers and thus rational.

9 Suppose $a, b,$ and c are odd integers and the equation $ax^2+bx+c=0$ has an integer solution.

Case 1: x is even.

Since the product of an even integers with any integer is even, $ax^2$and bx are both even. Since the sum of two even integers is even, $ax^2+ bx$ is also even. Since, c is an odd integer and the sum of an even integer and an odd integer is odd, ax^2+bx+c is odd.

Case 2: x is odd.

Since $a, b,$ and c are odd integers and the product of odd integers is odd, $ax^2$and bx are both odd. Since the sum of two odd integers is even, $ax^2$and bx are both even. Since, c is an odd integer and the sum of an even integer and an odd integer is odd, ax^2+bx+c is odd.

Thus, in either case, ax^2+bx+c is odd and hence cannot be equal to 0. Therefore the equation as no integer solutions.

10 $3(x-2)-4(3x+5)+2x-2=-7(x+4)$

$\Leftrightarrow 3x-6-12x-20+2x-2=-7(x+4)$

$\Leftrightarrow -7x-28=-7(x+4)$

$\Leftrightarrow -7(x+4)=-7(x+4)$

QED

11 a $\frac{x-2}{x}\div\frac{3x-6}{x^2+x}=\frac{x+1}{3}$

$\Leftrightarrow \frac{x-2}{x}\cdot\frac{x^2+x}{3x-6}=\frac{x+1}{3}$

$\Leftrightarrow \frac{x-2}{x}\cdot\frac{x(x+1)}{3(x-2)}=\frac{x+1}{3}$

$\Leftrightarrow \frac{1}{1}\cdot\frac{(x+1)}{3}=\frac{x+1}{3}$

$\Leftrightarrow \frac{x+1}{3}=\frac{x+1}{3}$

QED

b $x\neq 0,-1,2.$

12 Assume n and m are integers.

$18n + 6m = 1$

$\Rightarrow 3n + m = \frac{1}{6}$

Since the sum of two integers cannot be between 0 and 1, this is a contradiction and therefore n and m cannot be integers.

13 Suppose there exists two integers a and b, such that $a^2 - 4b = 2$.

$a^2 = 4b + 2$

$a^2 = 2(2b + 1)$

$\therefore a^2$ is even and a is also even.

$\therefore a = 2k$ for some integer k.

$a^2 - 4b = (2k)^2 - 4b = 2$

$4k^2 - 4b = 2$

$2k^2 - 2b = 1$

$2(k^2 - b) = 1$

Since $k^2 - b$ is an integer, 1 is therefore even.

Since this is not true, then $a^2 - 4b \neq 2$ by contradiction.

14 Find a counterexample that disproves the statement "For all $n \in \mathbb{R}$, $n^2 + 4 > 5$.

If $n = 0$ then $0 + 4 > 5$, which is false.

15 If $n = 7$:

$7! > 3^7$

$\Leftrightarrow 5040 > 2187$

Which is true.

Assume true for $n = k$:

$k! > 3^k$

Consider $n = k + 1$:

$(k + 1)! = (k + 1)k!$

$> (k + 1)3^k$

$> 3 \cdot 3^k$ (as $k > 6$)

$= 3^{k+1}$

Hence, if true for $n = k$ then also true for $n = k + 1$.

As true for $n = 7$, so true for all $n \geq 7$ by induction.

16 If $n = 1$:

$2! \geq 2^1(1!)^2$

$\Leftrightarrow 2 \geq 2$

Which is true.

Assume true for $n = k$:

$(2k)! \geq 2^k(k!)^2$

Consider $n = k + 1$:

$\big(2(k+1)\big)! = (2k+2)(2k+1)(2k)!$

$\Rightarrow \big(2(k+1)\big)! \geq (2k+2)(2k+1)(k!)^2 2^k$

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2$ since $2k + 1 > k + 1$

$= 2^{k+1}((k+1)!)^2$

Hence, if true for $n = k$ then also true for $n = k + 1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

17 If $n = 1$:

$4^1 + 2 = 6 = 3(2)$

Which is divisible by 3.

Assume true for $n = k$:

$4^k + 2 = 3j$, for some integer j

Consider $n = k + 1$:

$\Leftrightarrow 4^{k+1} + 2 = 4 \cdot 4^k + 2$

$\Leftrightarrow 4^{k+1} + 2 = 4(3j - 2) + 2$

$\Leftrightarrow 4^{k+1} + 2 = 12j - 8 + 2$

$\Leftrightarrow 4^{k+1} + 2 = 12j - 6$

$\Leftrightarrow 4^{k+1} + 2 = 3(4j - 2)$

$\Leftrightarrow 4^{k+1} + 2$ is divisible by 3.

Hence, if true for $n = k$ then also true for $n = k + 1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

18 If $n = 1$:

$1^3 + 11(1) = 12 = 3(4)$

Which is divisible by 3.

Assume true for $n = k$:

$k^3 + 11k = 3j$, for some integer j

Consider $n = k + 1$:

$(k + 1)^3 + 11(k + 1) = k^3 + 3k^2 + 3k + 11k + 11$

$(k + 1)^3 + 11(k + 1) = k^3 + 11 + 3k^2 + 3k$

$(k + 1)^3 + 11(k + 1) = 3j + 3k^2 + 3k$

$(k + 1)^3 + 11(k + 1) = 3(j + k^2 + k)$

$(k + 1)^3 + 11(k + 1)$ is divisible by 3.

Hence, if true for $n = k$ then also true for $n = k + 1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

19 If $n = 1$:

$1(1^2 + 5) = 6 = 6(1)$

Which is divisible by 6.

Assume true for $n = k$:

$k(k^2 + 5) = 6j$, for some integer j.

Consider $n = k + 1$:

$(k + 1)((k + 1)^2 + 5) = (k + 1)(k^2 + 2k + 6)$

$(k + 1)((k + 1)^2 + 5) = k^3 + 2k^2 + 6k + k^2 + 2k + 6$

$(k + 1)((k + 1)^2 + 5) = k^3 + 3k^2 + 8k + 6$

$(k + 1)((k + 1)^2 + 5) = (k^3 + 5k) + (3k^2 + 3k + 6)$

$(k+1)((k+1)^2+5) = k(k^2+5) + (3k^2+3k+6)$

$(k+1)((k+1)^2+5) = 6j + 3k(k+1) + 6$

Since $k(k+1)$ is even, all three terms are divisible by 6.

Hence, if true for $n = k$ then also true for $n = k+1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

20 If $n = 1$:

$1^2 = \frac{1(1+1)(2(1)+1)}{6}$

$\Leftrightarrow 1 = \frac{1(2)(3)}{6}$

$\Leftrightarrow 1 = \frac{6}{6}$

$\Leftrightarrow 1 = 1$

Which is true.

Assume true for $n = k$:

$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

Consider $n = k+1$:

We need to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$.

$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$

$\Leftrightarrow \sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$\Leftrightarrow \sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$

$\Leftrightarrow \sum_{i=1}^{k+1} i^2 = \frac{k(2k^2+3k+1)+6(k^2+2k+1)}{6}$

$\Leftrightarrow \sum_{i=1}^{k+1} i^2 = \frac{2k^3+3k^2+k+6k^2+12k+6}{6}$

$\Leftrightarrow \sum_{i=1}^{k+1} i^2 = \frac{2k^3+9k^2+k+13k+6}{6}$

$\Leftrightarrow \sum_{i=1}^{k+1} i^2 = \frac{(2k+3)(k+1)(k+2)}{6}$

QED

Hence, if true for $n = k$ then also true for $n = k+1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

21 If $n = 1$:

$\sum_{i=1}^{1} \frac{1}{(3(1)-1)(3(1)+2)} = \frac{1}{6(1)+4}$

$\Leftrightarrow \sum_{i=1}^{1} \frac{1}{(3-1)(3+2)} = \frac{1}{6+4}$

$\Leftrightarrow \sum_{i=1}^{1} \frac{1}{(2)(5)} = \frac{1}{10}$

$\Leftrightarrow \sum_{i=1}^{1} \frac{1}{10} = \frac{1}{10}$

Which is true.

Assume true for $n = k$:

$\sum_{i=1}^{k} \frac{1}{(3i-1)(3i+2)} = \frac{k}{6k+4}$

Consider $n = k + 1$:

We want to show that

$\sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \frac{k+1}{6(k+1)+4} = \frac{k+1}{6k+10} = \frac{k+1}{2(3k+5)}$

$\sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \sum_{i=1}^{k} \frac{1}{(3i-1)(3i+2)} + \frac{1}{(3(k+1)-1)(3(k+1)+2)}$

$\Leftrightarrow \sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \frac{k}{6k+4} + \frac{1}{(3k+3-1)(3k+3+2)}$

$\Leftrightarrow \sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$

$\Leftrightarrow \sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \frac{k(3k+5)}{2(3k+2)(3k+5)} + \frac{2}{2(3k+2)(3k+5)}$

$\Leftrightarrow \sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \frac{3k^2+5k+2}{2(3k+2)(3k+5)}$

$\Leftrightarrow \sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}$

$\Leftrightarrow \sum_{i=1}^{k+1} \frac{1}{(3i-1)(3i+2)} = \frac{k+1}{2(3k+5)}$

Hence, if true for $n = k$ then also true for $n = k + 1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

22 A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for all $n \in \mathbb{Z}^+$. Prove that

$u_n = 2^n - 1$.

If $n = 2$:

$u_2 = 2(u_1) + 1 = 3$

$2^2 - 1 = 3$

Assume true for $n = k$:

$u_k = 2^k - 1$

Consider $n = k + 1$:

We want to prove that $u_{k+1} = 2^{k+1} - 1$.

$u_{k+1} = 2u_k + 1$

$\Rightarrow u_{k+1} = 2\left(2^k - 1\right) + 1$

$\Rightarrow u_{k+1} = 2^{k+1} - 2 + 1$

$\Rightarrow u_{k+1} = 2^{k+1} - 1$

Hence, if true for $n = k$ then also true for $n = k + 1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

23 We want to prove that $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$

If $n = 1$:

$a = \frac{a(1-r^1)}{1-r}$

$\Leftrightarrow a = a$

Which is true.

Assume true for $n = k$:

$a + ar + ar^2 + ar^3 + \cdots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$

Consider $n = k + 1$:

$a + ar + ar^2 + ar^3 + \cdots + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$

$= \frac{a(1-r^k)}{1-r} + \frac{ar^k(1-r)}{1-r}$

$= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r}$

$= \frac{a - ar^{k+1}}{1-r}$

$= \frac{a(1-r^{k+1})}{1-r}$

Hence, if true for $n = k$ then also true for $n = k + 1$.

As true for $n = 1$, so true for all $n \geq 1, n \in \mathbb{Z}^+$.

1.4 Counting principles and the binomial theorem

1 Simplify then evaluate the following expressions:

a $\frac{8!}{6!}$

b $\frac{2!}{4!}$

c $\frac{4!2!}{6!}$

2 Simplify the following expressions:

a $\frac{n!}{(n-2)!}$

b $\frac{(n+2)!}{(n-1)!}$

c $\frac{(n^2-4)!}{(n-2)(n^2-5)!}$

d $\frac{(n!)^2}{(n-1)!(n+1)!}$

3 How many different odd 3 digit numbers are there?

4 The letters of the word BARRIER are arranged. How many different permutations are there, starting with:

a the letter R

b all three Rs?

5 Nghia has a bunch of identical toy cars: 4 are red, 3 are green and 2 are yellow. In how many ways can he line up his cars for a race?

6 Write down an English word where the number of possible permutations of the letters is equal to $\frac{5!}{2!}$. Explain your answer.

7 There are 5 people who need to sit in a row of chairs. Calculate the number of permutations for each situation below:

a There are 5 chairs.

b There are 6 chairs.

c There are 5 chairs, but Jack, one of the people, needs to be on either end.

d There are 5 chairs, but Jack needs to be in the middle.

e There are 5 chairs, but Jack and Jill, another one of the people, must sit together.

f There are 5 chairs, but Jack and Jill refuse to sit together.

8 Using only the digits 2, 3, 5, 6, 8, and 9 without repetition,

a How many 3 digit numbers greater than 6500 can be formed?

b How many 3 digit even numbers can be formed?

c How many numbers less than 400 can be formed?

9 Solve the following equations algebraically

a ${}_{n}P_{2} = 30$

b ${}_{n}P_{3} = 60$

c $2n + {}_{n}P_{2} = 56$

10 A jury of 6 men and 6 women must be chosen from a pool of 14 men and 16 women. How many different juries can be formed?

11 Consider the word CABBAGE. How many permutations are there if:

a All the letters are used?

b All letters are used and it starts with a B?

c All letters are used and it starts with a vowel?

12 From a group of 5 students, three must be selected to work on a committee.

a How many different committees are possible?

b How many different committees are possible if Mohammed, one of the students, must be on the committee?

c How many different committees are possible if Mohammed and Tenzin, another one of the students, cannot work together?

13 Find the indicated term in the expansions below.

a The 6$^{\text{th}}$ term of $(2x - y)^9$.

b The 4$^{\text{th}}$ term of $(x + 5y)^7$.

c The middle term of $(x^2 - 2)^8$.

d The constant term of $(3x - 2)^{10}$.

14

a Find the term containing x^3 in the expansion of $(x - 3)^8$.

b Hence, find the term containing x^4 in the expansion of $-2x(x - 3)^8$.

15 The fourth term in the expansion of $(2x - k)^8$ is $-387072x^5$. Find the value of k.

16 Find the coefficient of the term containing a^3b^6 in the expansion of $(a - b^2)^6$.

17 Use the binomial theorem to estimate 1.81^4 to three decimal places.

18 Expand $\frac{3}{(1-2x)}$ for $|x| < \frac{1}{2}$.

Answers

1 **a** $\frac{8\times7\times6!}{6!}=56$

b $\frac{2!}{4\times3\times2!}=\frac{1}{12}$

c $\frac{4!\times2!}{6\times5\times4!}=\frac{2\times1}{6\times5}=\frac{1}{15}$

2 **a** $\frac{n!}{(n-2)!}=\frac{n(n-1)(n-2)!}{(n-2)!}=n(n-1)=n^2-n$

b $\frac{(n+2)!}{(n-1)!}=\frac{(n+2)(n+1)n(n-1)!}{(n-1)!}=n(n^2+3n+2)=n^3+3n^2+2n$

c $\frac{(n^2-4)!}{(n-2)(n^2-5)!}=\frac{(n^2-4)(n^2-5)!}{(n-2)(n^2-5)!}=\frac{n^2-4}{n-2}=\frac{(n-2)(n+2)}{(n-2)}=n+2$

d $\frac{(n!)^2}{(n-1)!(n+1)!}=\frac{n!n!}{(n-1)!(n+1)!}=\frac{n!}{(n-1)!}\cdot\frac{n!}{(n+1)!}=\frac{n(n-1)!}{(n-1)!}\cdot\frac{n!}{(n+1)n!}=n\cdot\frac{1}{(n+1)}=\frac{n}{n+1}$

3 An odd three digit cannot start with 0 and must end with one of 1, 3, 5, 7, 9, therefore

$(9)(10)(5)=450$

4 **a** $\frac{3(6!)}{3!}=360$

b $\frac{3!(4!)}{3!}=24$

5 $\frac{9!}{4!3!2!}=\frac{362880}{(24)(6)(2)}=\frac{362880}{288}=1260$

6 Answers will vary. Some possible answers are: hello, apple, needy.

7 **a** ${}_5P_5=120$

b ${}_6P_5=720$

c $2(4!)=48$

d $(4!)=24$

e $2(4!)=48$

f $5!-2(4!)=120-48=72$

8 **a** Case 1: Start with a 6 with second digit 5, 8, or 9

$(1)(3)(4)(3)=36$

Case 2: Starting with a 8 or 9

$(2)(5)(4)(3)=120$

Total: $36+120=156$

b An even number must end in 2, 6, or 8:

$(5)(4)(3) = 60$

c Case 1: three digit number

$(2)(5)(4) = 40$

Case 2: two digit number

$(6)(5) = 30$

Case 1: one digit number

6

Total: $40 + 30 + 6 = 76$

9 **a** ${}_{\mathrm{n}}P_2 = 30$

$\frac{n!}{(n-2)!} = 30$

$n(n-1) = 30$

$n^2 - n - 30 = 0$

$(n-6)(n+5) = 0$

$n = 6, -5$

Since $n \geq 0, n = 6$

b ${}_{\mathrm{n}}P_3 = 60$

$\frac{n!}{(n-3)!} = 60$

$n(n-1)(n-2) = 60$

$n(n^2 - 3n + 2) = 60$

$n^3 - 3n^2 + 2n - 60 = 0$

By GDC, $n = 5$

c 2n + ${}_{\mathrm{n}}P_2 = 56$

$2n + \frac{n!}{(n-2)!} = 56$

$2n + n(n-1) = 56$

$n^2 + n - 56 = 0$

$(n+8)(n-7) = 0$

$n = 7, -8$

Since $n \geq 0, n = 7$

10 $\binom{14}{6} \cdot \binom{16}{6} = 24\,048\,024$

11 a $\frac{7!}{2!2!} = \frac{5040}{4} = 1260$

b $\frac{(2)(6!)}{2!2!} = \frac{1440}{4} = 360$

c $\frac{(3)(6!)}{2!2!} = \frac{2160}{4} = 540$

12 a $\binom{5}{3} = 10$

b $\binom{1}{1} \cdot \binom{4}{2} = 6$

c Case 1: Mohammed is on the committee, but Tenzin is not.

$\binom{1}{1} \cdot \binom{3}{2} = 3$

Case 2: Tenzin is on the committee, but Mohammed is not.

$\binom{1}{1} \cdot \binom{3}{2} = 3$

Case 3: Neither one is on the committee.

$\binom{3}{3} = 1$

Total: 7

13 a $\binom{9}{5}(2x)^4(-y)^5 = 126(16x^4)(-y^5) = -2016x^4y^5$

b $\binom{7}{3}(x)^4(5y)^3 = 35(x^4)(125y^3) = 4375x^4y^3$

c There are 9 terms, so the middle term is the 5th.

$\binom{8}{4}(x^2)^4(-2)^4 = 70(x^8)(16) = 1120x^8$

d $\binom{10}{10}(3x)^0(-2)^{10} = 1024$

14 a $\binom{8}{5}(x)^3(-3)^5 = 56(x^3)(-243) = -13608x^3$

b $(-2x)(-13608x^3) = 27216x^4$

15 $\binom{8}{3}(2x)^5(-k)^3 = -387072x^5$

$56(32x^5)(-k^3) = -387072x^5$

$-1792k^3x^5 = -387072x^5$

$-1792k^3 = -387072$

$k^3 = 216$

$\sqrt[3]{k^3} = \sqrt[3]{216}$

$k = 6$

16 $\binom{6}{3}(a)^3(-b^2)^3 = 20(a^3)(-b^6) = -20a^3b^6$

The coefficient is -20.

17

$1.81^4 = (1 + 0.81)^4 =$
$\binom{4}{0}(1)^4(0.81)^0 + \binom{4}{1}(1)^3(0.81)^1 + \binom{4}{2}(1)^2(0.81)^2 + \binom{4}{3}(1)^1(0.81)^3 + \binom{4}{4}(1)^0(0.81)^4$

$1.81^4 = 1 + 4(0.81) + 6(0.81^2) + 4(0.81^3) + (0.81)^4$

$1.81^4 = 10.7328 \approx 10.733$

18 $\frac{3}{(1-2x)} = 3(1 - 2x)^{-1}$

$\frac{3}{(1-2x)} = 3\left(1 + (1)(2x) + \frac{(1)(2)}{2!}(2x)^2 + \frac{(1)(2)(3)}{3!}(2x)^3 + \cdots\right)$

$\frac{3}{(1-2x)} = 3(1 + 2x + 4x^2 + 8x^3 + \cdots)$

$\frac{3}{(1-2x)} = 3 + 6x + 12x^2 + 24x^3 + \cdots$

2.1 Functional relationships

1 For each relation below, state its domain and range and if it is a function or not.

a $y=-3x+2$ **b** $y=4$ **c** $\{(2,3),(3,9),(3,6),(5,-1)\}$

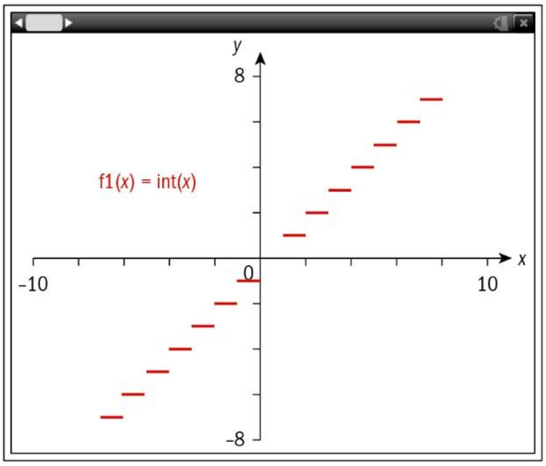
d

x	-1	3	8	11	16
y	-2	-2	-2	-2	8

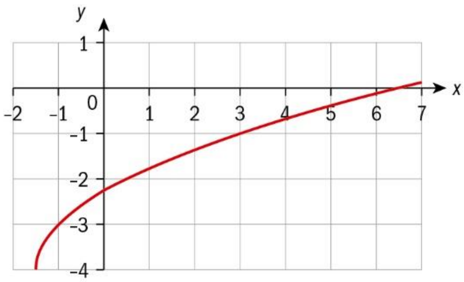
e All multiplies of 3 greater than 9.

2 For each graph below, state if it is a function or not and write down its domain and range.

a



b



Answers

1 **a** Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $y \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Function: yes

b Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $\{4\}$

Function: yes

c Domain: $\{2, 3, 5\}$

Range: $\{-1, 3, 6, 9\}$

Function: no

d Domain: $\{-1, 3, 8, 11, 16\}$

Range: $\{-2, 8\}$

Function: yes

e Domain: $x > 3$, $x \in \mathbb{Z}^+$

Range: $\{12, 15, 18, \ldots\}$

Function: yes

2 **a** Function: yes

Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $y \in \mathbb{Z}^+$

b Function: yes

Domain: $x \geq -1.5$ or $[-1.5, \infty)$ or $[-1.5, \infty[$

Range: $y \geq -4$ or $[-4, \infty)$ or $[-4, \infty[$

2.2 Special functions and their graphs

1 State the concavity of $f(x) = -\frac{1}{2}x^2 + 4x - 1$ and find the:

a Equation of the axis of symmetry

b Vertex

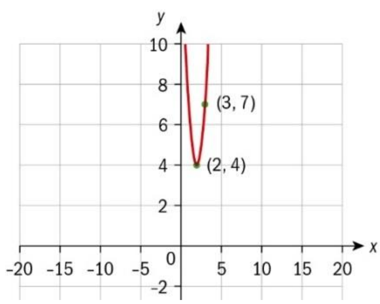
c Domain and range

d Vertex form

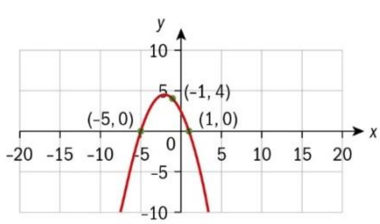
2 Find the equation of the quadratic equation that has a vertex at $(-2, 5)$ and a y-intercept of $(0, -2)$. State your answer in standard form.

3 Write the quadratic equation for each graph below.

a



b



4 In a 110 volt electrical circuit having a resistance of 144 ohms, the available power P in watts is a function of I, the amount of current flowing in amperes. If $P = 110I - 144I^2$, how many amperes will produce the maximum power in the circuit? What is the maximum power?

5 For the function $y = \frac{3x-1}{4-2x}$,

a **i** State the equation of all asymptotes.

ii Write down the domain and range.

b Hence, sketch the graph.

6 **a** Determine the equations of the asymptotes for $y = \dfrac{-1}{6x^2 - 5x + 1}$.

b Use your GDC to sketch the graph.

c Write down the domain and range.

7 Determine the domain and range of the following functions and state any asymptotes. Confirm your answers graphically.

a $y + 1 = \sqrt{2x - 4}$ **b** $y = \dfrac{-2}{\sqrt{x^2 - 7x + 12}}$ **c** $y = \dfrac{4}{x^2 - 9}$

8 Express the following expressions in partial fractions

a $\dfrac{3x + 2}{x^2 + x}$ **b** $\dfrac{3x + 11}{x^2 - x - 6}$ **c** $\dfrac{5x - 4}{x^2 - x - 2}$

9 Sketch the graphs of the following absolute value functions and state the domain and range.

a $y = 2|x - 4| + 3$ **b** $y = |2(x - 1)^2 + 3|$ **c** $y = |-2x^2 + 6x - 1|$

10 Solve the following equations. Check your answers either algebraically or graphically.

a $|x^2 - 2x - 5| = 2$ **b** $|x^2 - 9| = x + 3$

c $|2a - 1| = -a + 4$ **d** $|x^2 - 4x| = 0$

11 Solve the following inequalities. Justify your answers graphically.

a $2|3x - 9| < 36$ **b** $|5x + 6| \leq 1$

c $|3x - 2| + 9 > 5$ **d** $|x^2 - 3x + 1| < 1$

12 Solve $\left|\dfrac{2}{x + 2}\right| \geq 4$ algebraically.

13 Solve the following inequalities graphically.

a $\left|\dfrac{x - 6}{x - 2}\right| \leq 4$ **b** $-2 \geq -3 + \left|\dfrac{2}{x - 1}\right|$

14 Consider the function $f(x) = \begin{cases} -x + 2, & x < 1 \\ 3, & 1 \leq x \leq 2 \\ x^2, & x > 2 \end{cases}$

a Evaluate $f(-1)$, $f(0)$, $f(1)$, $f(2)$ and $f(4)$

b Sketch the graph of $f(x)$

c Write the domain and range of $f(x)$

15 Rewrite $g(x) = |-2x + 6| + 2$ as a piecewise function.

Answers

1 Concave down

a $x = \frac{-b}{2a} = \frac{-4}{2\times\left(-\frac{1}{2}\right)} = \frac{-4}{-1} = 4$

b $f(4) = -\frac{1}{2}\times 4^2 + 4\times 4 - 1 = -\frac{1}{2}\times 16 + 16 - 1 = -8 + 18 - 1 = 7$, vertex: $(4, 7)$

c Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $y \le 7$ or $(-\infty, 7]$ or $]-\infty, 7]$

d $f(x) = -\frac{1}{2}(x-4)^2 + 7$

2

$$\begin{aligned} y &= a(x+2)^2 + 5 \\ -2 &= a(0+2)^2 + 5 \\ -7 &= 4a \\ a &= -\frac{7}{4} \\ y &= -\frac{7}{4}(x+2)^2 + 5 \end{aligned}$$

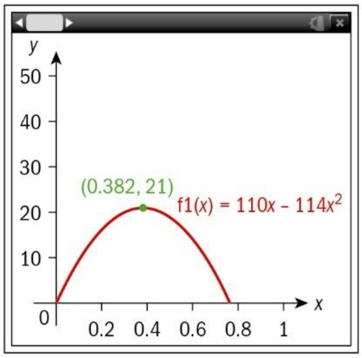
3 a

$$\begin{aligned} y &= a(x-2)^2 + 4 \\ 7 &= a(3-2)^2 + 4 \\ 3 &= a \\ y &= 3(x-2)^2 + 4 \end{aligned}$$

b

$$\begin{aligned} y &= a(x-1)(x+5) \\ 4 &= a(-1-1)(-1+5) \\ 4 &= -8a \\ a &= -\frac{1}{2} \\ y &= -\frac{1}{2}(x-1)(x+5) \end{aligned}$$

4 $I = \frac{-b}{2a} = \frac{-110}{2\times(-144)} = 0.382$, $P = 110\times 0.382 - 144\times 0.382^2 = 21$



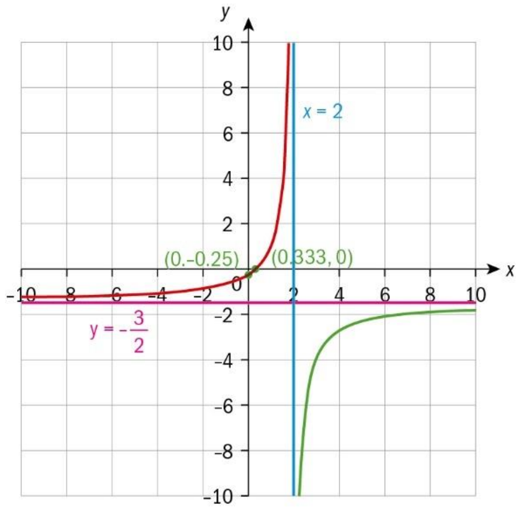
0.382 amperes will produce a maximum power of 21 watts.

5 **a** **i** Vertical asymptote: $x = 2$, horizontal asymptote: $y = -\frac{3}{2}$

ii Domain: $x \in \mathbb{R}$, $x \neq 2$ or $(-\infty, 2) \cup (2, \infty)$ or $]-\infty, 2[\cup]2, \infty[$

Range: $y \in \mathbb{R}$, $y \neq -\frac{3}{2}$ or $\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$ or $\left]-\infty, -\frac{3}{2}\right[\cup \left]-\frac{3}{2}, \infty\right[$

b



6 **a** Vertical asymptote where $6x^2 - 5x + 1 = 0 \Rightarrow (3x-1)(2x-1) = 0$: $x = \frac{1}{3}$ and $x = \frac{1}{2}$

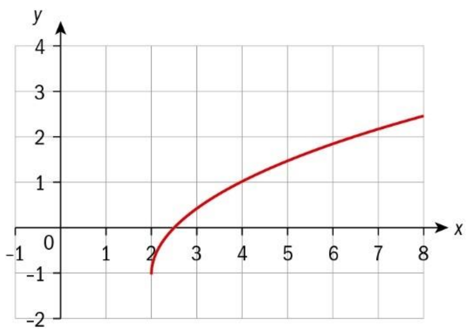
Horizontal asymptote: $y = 0$

c Domain: $x \in \mathbb{R}$, $x \neq \frac{1}{3}$, $x \neq \frac{1}{2}$ or $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ or $\left]-\infty, \frac{1}{3}\right[\cup \left]\frac{1}{3}, \frac{1}{2}\right[\cup \left]\frac{1}{2}, \infty\right[$

Range: $y < 0$, $y \geq 24$ or $(-\infty, 0) \cup [24, \infty)$ or $]-\infty, 0[\cup [24, \infty[$

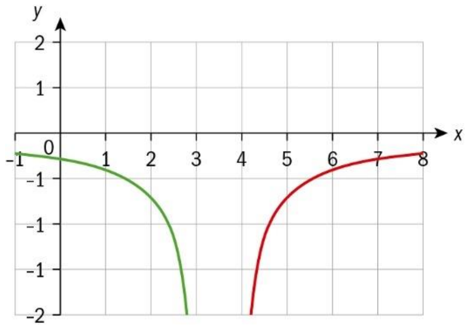
7 **a** Domain: $x \geq 2$ or $[2, \infty)$ or $[2, \infty[$, range: $y \geq -1$ or $[-1, \infty)$ or $[-1, \infty[$

Asymptotes: none



b Domain: $x < 3$, $x > 4$ or $(-\infty, 3) \cup (4, \infty)$ or $]-\infty, 3[\cup]4, \infty[$, range: $y < 0$ or $(-\infty, 0)$ or $]-\infty, 0[$

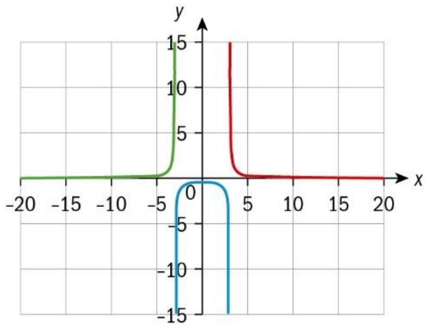
Asymptotes: $x = 3$ and $x = 4$



c Domain: $x \in \mathbb{R}$, $x \neq -3$, $x \neq 3$ or $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ or $]-\infty,-3[\cup]-3,3[\cup]3,\infty[$

Range: $y \le -\frac{4}{9}$, $y > 0$ or $\left(-\infty,-\frac{4}{9}\right)\cup(0,\infty)$ or $\left]-\infty,-\frac{4}{9}\right[\cup\,]0,\infty[$

Asymptotes: $x = -3$ and $x = 3$

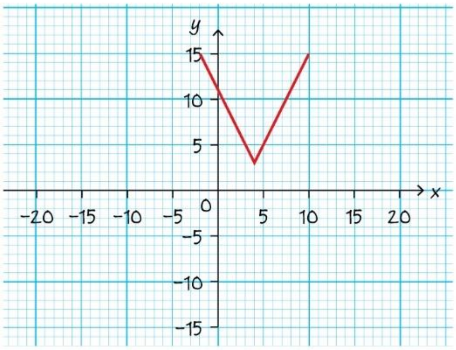


8 **a** $\frac{3x+2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} \therefore 3x+2 = A(x+1)+Bx$. Let $x=-1$:
$3\times(-1)+2 = A\times(-1+1)+B\times(-1) \therefore B=1$. Let $x=0$: $3\times 0+2 = A\times(0+1)+B\times 0 \therefore A=2$. So
$\frac{3x+2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} = \frac{2}{x} + \frac{1}{x+1}$

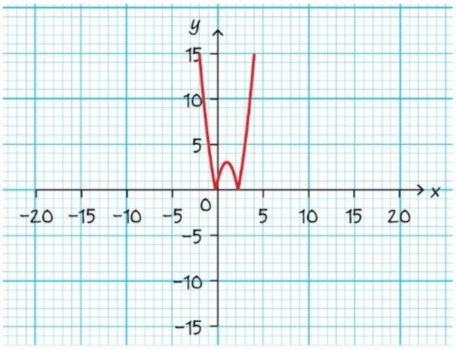
b $\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} \therefore 3x+11 = A(x+2)+B(x-3)$. Let $x=-2$:
$3\times(-2)+1 = A\times(-2+2)+B\times(-2-3) \therefore B=-1$. Let $x=-3$:
$3\times 3+11 = A\times(3+2)+B\times(3-3) \therefore A=4$. So $\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{4}{x-3} - \frac{1}{x+2}$

c $\frac{5x-4}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1} \therefore 5x-4 = A(x+1)+B(x-2)$. Let $x=2$:
$5\times 2-4 = A\times(2+1)+B\times(2-2) \therefore A=2$. Let $x=-1$:
$5\times(-1)-4 = A\times(-1+1)+B\times(-1-2) \therefore B=3$. So $\frac{5x-4}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{2}{x-2} + \frac{3}{x+1}$

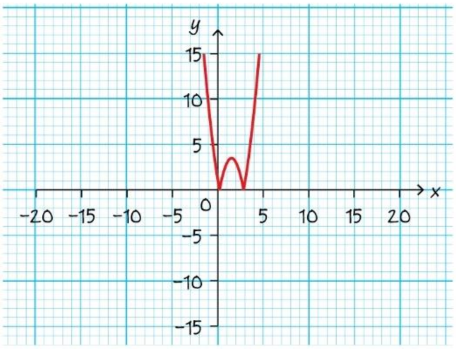
9 **a** Domain: $x \in \mathbb{R}$ or $(-\infty,\infty)$ $]-\infty,\infty[$, range: $y \ge 3$ or $[3,\infty)$ or $[3,\infty[$



b Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ $]-\infty, \infty[$, range: $y \geq 0$ or $[0, \infty)$ or $[0, \infty[$



c Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ $]-\infty, \infty[$, range: $y \geq 0$ or $[0, \infty)$ or $[0, \infty[$



10 a Case 1: $x^2 - 2x - 5 = 2$

$$0 = x^2 - 2x - 7$$

$$x = \frac{2 \pm \sqrt{2^2 - 4 \times 1 \times (-7)}}{2 \times 1}$$

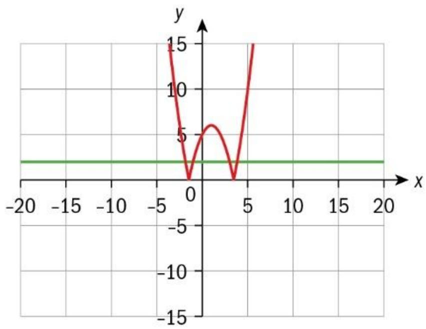
$$x = 1 \pm 2\sqrt{2}$$

Case 2: $-x^2 + 2x + 5 = 2$

$$\begin{aligned} 0 &= -x^2 + 2x + 3 \\ &= x^2 - 2x - 3 \\ &= (x-3)(x+1) \end{aligned}$$

$$x = -1, 3$$

Check:



b Case 1: $x^2 - 9 = x + 3$

$$\begin{aligned} 0 &= x^2 - x - 12 \\ &= (x-4)(x+3) \\ x &= -3, 4 \end{aligned}$$

Case 2: $-x^2 + 9 = x + 3$

$$\begin{aligned} 0 &= -x^2 - x + 6 \\ &= x^2 + x - 6 \\ &= (x-2)(x+3) \\ x &= -3, 2 \end{aligned}$$

Check:

$$\begin{aligned} \left|(-3)^2 - 9\right| &= -3 + 3 \\ |9 - 9| &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} \left|2^2 - 9\right| &= 2 + 3 \\ |4 - 9| &= 5 \\ 5 &= 5 \end{aligned}$$

$$\begin{aligned} \left|4^2 - 9\right| &= 4 + 3 \\ |16 - 9| &= 7 \\ 7 &= 7 \end{aligned}$$

c Case 1: $2a - 1 = -a + 4$

$$\begin{aligned} 3a &= 5 \\ a &= \frac{5}{3} \end{aligned}$$

Case 2: $-2a + 1 = -a + 4$

$$\begin{aligned} -a &= 3 \\ a &= -3 \end{aligned}$$

Check:

$$\begin{aligned} \left|2 \times \frac{5}{3} - 1\right| &= -\frac{5}{3} + 4 \\ \left|\frac{10}{3} - 1\right| &= \frac{7}{3} \\ \frac{7}{3} &= \frac{7}{3} \end{aligned}$$

$$\begin{aligned} |2 \times (-3) - 1| &= -(-3) + 4 \\ |-6 - 1| &= 7 \\ 7 &= 7 \end{aligned}$$

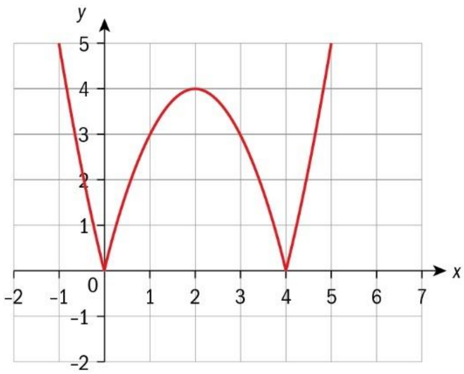
d Case 1: $x^2 - 4x = 0$

$$\begin{aligned} x(x-4) &= 0 \\ x &= 0, 4 \end{aligned}$$

Case 2: $-x^2 + 4x = 0$

$$\begin{aligned} x(-x+4) &= 0 \\ x &= 0, 4 \end{aligned}$$

Check:



11 a Case 1: $2(3x-9)<36$

$6x-18<36$
$6x<54$
$x<9$

Case 2: $2(-3x+9)<36$

$-6x+18<36$
$-6x<18$
$x>-3$

$\therefore -3<x<9$

b Case 1: $5x+6\le 1$

$5x\le -5$
$x\le -1$

Case 2: $-5x-6\le 1$

$-5x\le 7$
$x\ge -\dfrac{7}{5}$

$\therefore -\dfrac{7}{5}\le x\le -1$

c Case 1: $3x-2+9>5$

$3x>-2$
$x>-\dfrac{2}{3}$

Case 2: $-3x+2+9>5$

$-3x>-6$
$x<2$

$\therefore x\in\mathbb{R}$

d Case 1: $x^2-3x+1<1$

$x^2-3x<0$
$x(x-3)<0$
$0<x<3$

Case 2: $-x^2+3x-1<1$

$-x^2+3x-2<0$
$x^2-3x+2<0$
$(x-1)(x-2)<0$
$x<1$
$x>2$

$\therefore 0<x<1, 2<x<3$

12 Case 1: $\dfrac{2}{x+2}\ge 4$

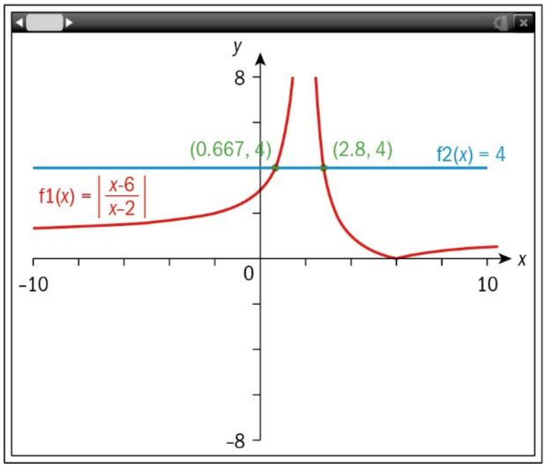
$4(x+2)\le 2$
$4x+8\le 2$
$4x\le -6$
$x\le -\dfrac{3}{2}$

Case 2: $-\dfrac{2}{x+2}\ge 4$

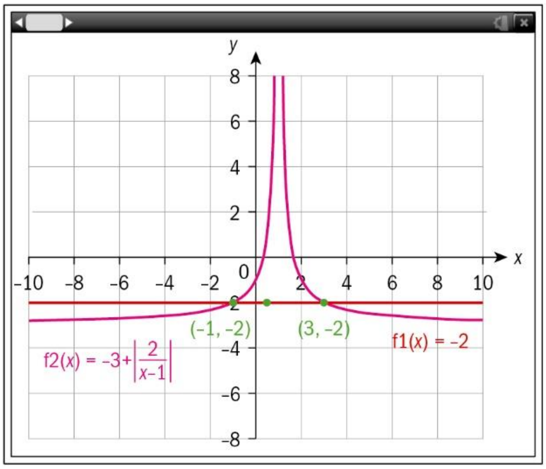
$4(x+2)\ge -2$
$4x+8\ge -2$
$4x\ge -10$
$x\ge -\dfrac{5}{2}$

$\therefore -\dfrac{5}{2}\le x\le -\dfrac{3}{2}, x\ne -2$

13 a $x \le 0.667$, $x \ge 2.8$

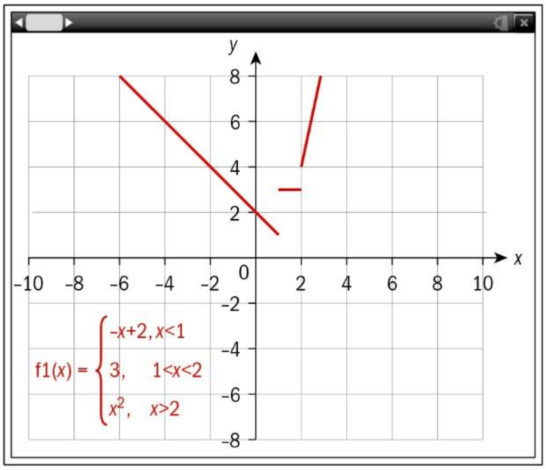


b $x \le -1$, $x \ge 3$



14 a $f(-1) = -(-1) + 2 = 3$, $f(0) = -0 + 2 = 2$, $f(1) = 3$, $f(2) = 3$, $f(4) = 4^2 = 16$

b



c Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ $]-\infty, \infty[$, range: $y > 1$ or $(1, \infty)$ or $]1, \infty[$

15 $g(x) = \begin{cases} -2x + 8, & x < 3 \\ 2x - 4, & x \ge 3 \end{cases}$ or $g(x) = \begin{cases} -2x + 8, & x \le 3 \\ 2x - 4, & x > 3 \end{cases}$

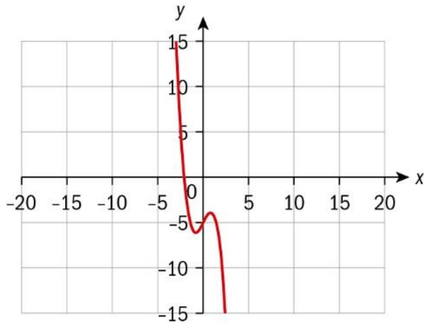
2.3 Classification of functions

1 Classify each function below as one-to-one or many-to-one. Justify your answers.

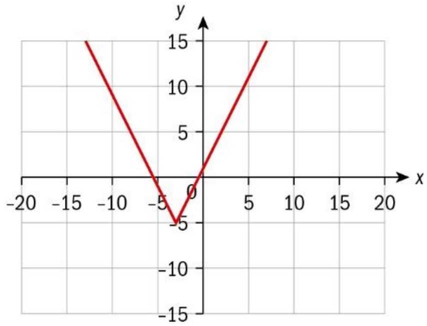
a $f(x) = x|x|$

b $y = 0.1x^2 + 2.3x + 1.4$

c



d



2 Determine algebraically whether the following functions are even, odd or neither. Verify your answers graphically.

a $f(x) = -2x^2 + 4$ **b** $g(x) = 2x^3 - 4x$ **c** $f(x) = -|x - 4|$

3 Show that $f(x) = \dfrac{x^3 - x}{x^2 + 1}$ is odd.

Answers

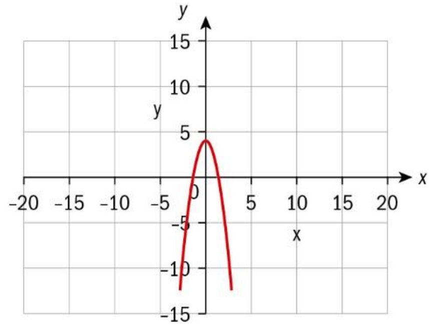
1 **a** One-to-one because it passes the vertical and horizontal line tests

b Many-to-one because it fails the horizontal line test

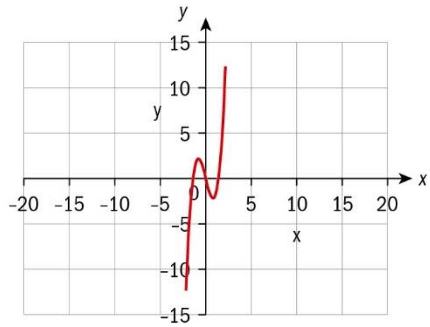
c One-to-one because it passes the vertical and horizontal line tests

d Many-to-one because it fails the horizontal line test

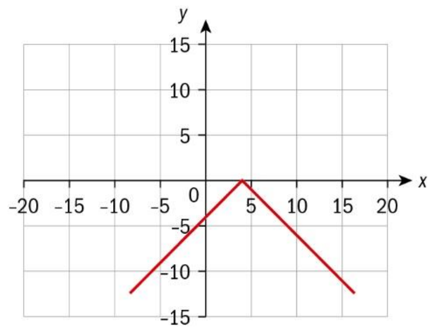
2 **a** $f(-x) = -2(-x)^2 + 4 = -2x^2 + 4 = f(x)$ so even



b $g(-x) = 2(-x)^3 - 4(-x) = -2x^3 + 4x = -(2x^3 - 4x) = -g(x)$ so odd



c $f(-x) = -|-x - 4|$ which is neither $-f(x)$ nor $f(x)$ so neither odd nor even



3 $-f(-x) = -\left(\dfrac{(-x)^3 - (-x)}{(-x)^2 + 1}\right) = -\left(\dfrac{-x^3 + x}{x^2 + 1}\right) = \dfrac{x^3 - x}{x^2 + 1} = f(x)$

2.4 Operations with functions

1 Given $f(x)=\sqrt{(x)}$ and $g(x)=x^2+3x$

a **i** Find a simplified expression for $g\circ f(x)$ and state the domain.

ii Find a simplified expression for $f\circ g(x)$ and state the domain.

b Hence, explain if $f(x)$ and $g(x)$ are inverses.

2 Given $f(x)=\dfrac{2}{x+1}$, $g(x)=2\sqrt{x}+3$ and $h(x)=4$:

a Evaluate

i $f(0)$ **ii** $f(-1)$ **iii** $g(9)$

iv $h(-1)$ **v** $g(f(7))$ **vi** $f(g(h(0)))$

b Find the domains of $f(x)$, $g(x)$ and $h(x)$.

c Hence state the domain of $f(g(h(x)))$.

3 You purchase a new refrigerator. You have no way to get it home, so you pay the delivery fee of $30. The sales tax is 5%.

a Write a function $f(x)$ that represents the cost of the fridge and the delivery fee.

b Write a function $g(x)$ that represents the cost of the fridge and the tax.

c Find and interpret both $f(g(x))$ and $f(g(x))$.

d If taxes cannot be charged on delivery, which composition of functions from part **c** should be used to calculate the total amount you must pay to the store?

4 **a** For each function below, sketch its graph.

b On the same set of axis, sketch the graph of its inverse.

c State if each function is one-to-one. Justify your answer.

i $f(x)=-3x+1$ **ii** $f:x\to -2x^2+4$ **iii** $g(x)=3$

5 **a** Find the inverse of $f(x)=\dfrac{1}{x+1}-2$, $x\neq -1$ algebraically.

b Sketch the graph of $f^{-1}(x)$.

c Hence, state the domain and range of $f^{-1}(x)$.

d Is $f(x)$ a one-to-one function? Justify your answer.

6 Determine algebraically of the following pairs of functions are inverses or not.

a $f(x) = \sqrt{3x - 2}$ and $g(x) = \dfrac{x^2}{3} + \dfrac{2}{3}$

b $g(x) = -\dfrac{3}{4}x + 5$ and $h(x) = -\dfrac{4x - 20}{3}$

7 Show that the function $f(x) = \dfrac{x}{x - 1}$ is a self-inverse.

8 Find the value of k such that $f : x \to \dfrac{3x - 5}{x + k}$ is a self-inverse function.

Answers

1 **a** **i** $g \circ f(x) = \left(\sqrt{x}\right)^2 + 3\sqrt{x} = x + 3\sqrt{x}$, domain: $x \geq 0$ or $[0, \infty)$ or $[0, \infty[$

ii $f \circ g(x) = \sqrt{x^2 + 3x}$, domain: $x \geq 0$ or $[0, \infty)$ or $[0, \infty[$

b Since $g \circ f(x) \neq f \circ g(x) \neq x$, they are not inverses.

2 **a** **i** $f(0) = \dfrac{2}{0+1} = 2$

ii $f(0) = \dfrac{2}{-1+1} = \dfrac{2}{0} = \text{undefined}$

iii $g(9) = 2\sqrt{9} + 3 = 2 \times 3 + 3 = 9$

iv $h(-1) = 4$

v $g(f(7)) = g\left(\dfrac{2}{7+1}\right) = g\left(\dfrac{2}{8}\right) = g\left(\dfrac{1}{4}\right) = 2\sqrt{\dfrac{1}{4}} + 3 = 2 \times \dfrac{1}{2} + 3 = 1 + 3 = 4$

vi $f(g(h(0))) = f(g(4)) = f\left(2\sqrt{4} + 3\right) = f(4+3) = f(7) = \dfrac{2}{7+1} = \dfrac{1}{4}$

b Domain of $f(x)$: $x \in \mathbb{R}$, $x \neq -1$

Domain of $g(x)$: $x \geq 0$

Domain of $h(x)$: $x \in \mathbb{R}$

c $x \geq 0$

3 **a** $f(x) = x + 30$

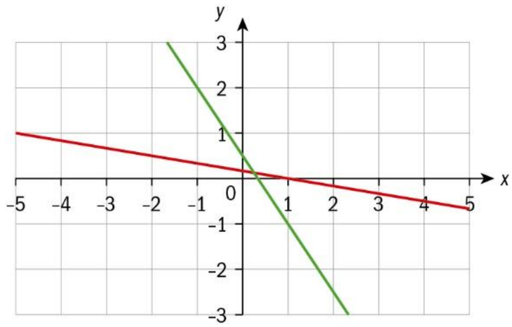
b $g(x) = 1.05x$

c $f(g(x))$ represents the cost of the refrigerator plus the tax and the delivery fee.

$g(f(x))$ represents paying the tax on the cost of the refrigerator and the delivery fee.

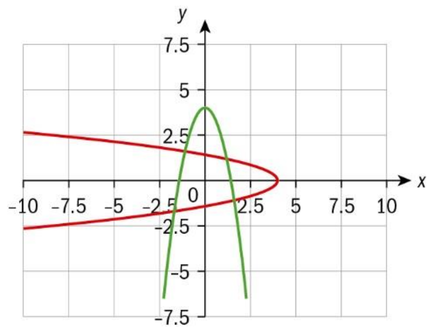
d $f(g(x))$

4 **i** **a**, **b**



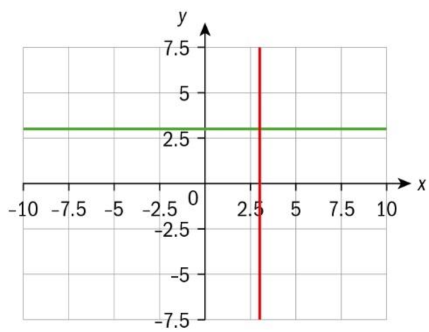
c Yes, both $f(x)$ and its inverse pass the vertical line test.

i **a**, **b**



c No, the inverse fails the vertical line test.

iii **a**, **b**



c No, the inverse fails the vertical line test.

5 **a** $f(x) = \dfrac{1}{x+1} - 2$

$$x = \frac{1}{y+1} - 2$$

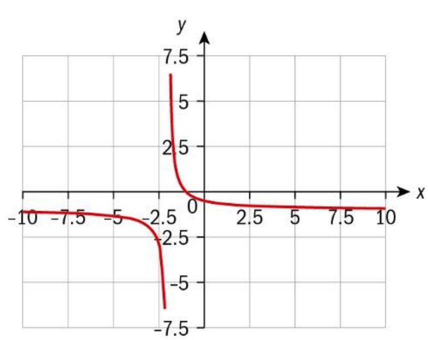
$$x + 2 = \frac{1}{y+1}$$

$$y + 1 = \frac{1}{x+2}$$

$$y = \frac{1}{x+2} - 1$$

$$f^{-1}(x) = \frac{1}{x+2} - 1$$

b



c Domain: $x \in \mathbb{R}$, $x \neq -2$, range: $y \in \mathbb{R}$, $y \neq -1$

d Yes. The inverse passes the vertical line test.

6 **a** $f(g(x)) = \sqrt{3\left(\frac{x^2}{3} + \frac{2}{3}\right) - 2} = \sqrt{x^2 + 2 - 2} = \sqrt{x^2} = x$

$$g(f(x)) = \frac{\left(\sqrt{3x-2}\right)^2}{3} + \frac{2}{3} = \frac{3x - 2 + 2}{3} = \frac{3x}{3} = x$$

Since $f(g(x)) = g(f(x)) = x$, these are inverses.

b $g(h(x)) = -\frac{3}{4}\left(-\frac{4x-20}{3}\right) + 5 = \frac{12x - 60}{12} + 5 = x - 5 + 5 = x$

$$h(g(x)) = -\frac{4\left(-\frac{3}{4}x + 5\right) - 20}{3} = -\frac{3x + 20 - 20}{3} = \frac{3x}{3} = x$$

Since $f(g(x)) = g(f(x)) = x$, these are inverses.

7 $f(f(x)) = \dfrac{\left(\frac{x}{x-1}\right)}{\left(\frac{x}{x-1}\right) - 1} = \dfrac{\left(\frac{x}{x-1}\right)}{\left(\frac{x}{x-1}\right) - \left(\frac{x-1}{x-1}\right)} = \dfrac{\left(\frac{x}{x-1}\right)}{\left(\frac{x - x + 1}{x-1}\right)} = \dfrac{\left(\frac{x}{x-1}\right)}{\left(\frac{1}{x-1}\right)} = x$

Therefore $f(x) = \dfrac{x}{x-1}$ is a self-inverse.

8 To be a self-inverse, $f(f(x)) = x$,

$$x = \frac{3\left(\frac{3x-5}{x+k}\right) - 5}{\left(\frac{3x-5}{x+k}\right) + k} = \frac{\left(\frac{9x-15}{x+k}\right) - 5}{\left(\frac{3x-5}{x+k}\right) + k} = \frac{\frac{(9x-15)}{(x+k)} - \frac{5(x+k)}{(x+k)}}{\frac{(3x-5)}{(x+k)} + \frac{k(x+k)}{(x+k)}} = \frac{\frac{9x - 15 - 5x - 5k}{x+k}}{\frac{3x - 5 + kx + k^2}{x+k}} = \frac{9x - 15 - 5x - 5k}{3x - 5 + kx + k^2} = x$$

So $4x - 5k + 15 = 3x^2 - 5x + kx^2 + k^2 = (3+k)x^2 + (k^2 - 5)x$, since there is no x^2 on the left hand side:

$3 + k = 0 \therefore k = -3$

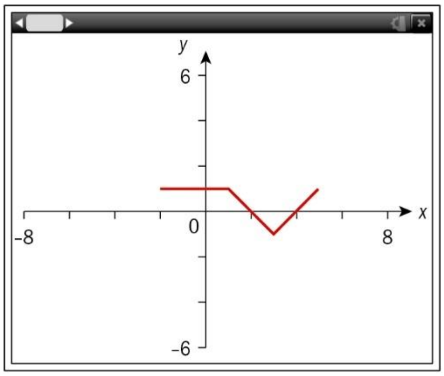
2.5 Function transformations

1 For each function below, sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.

a $y - 2 = x^2 + 4x$

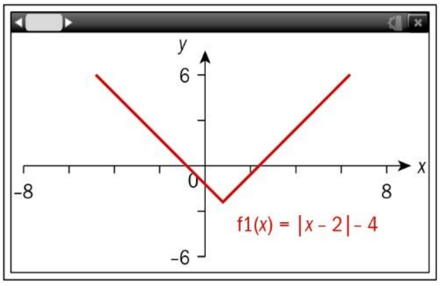
b $y = \sqrt{2x} - 2$

2 Given the graph of $y = g(x)$ below, sketch $y = |g(x)|$ and $y = g(|x|)$.



3 On the same set of axes, draw the graphs of $y = x^2 - 9$ and $y = \dfrac{1}{x^2 - 9}$. Label any y-intercepts and asymptotes.

4 Given the graph of $g(x)$ below, sketch the graph of $\dfrac{1}{g(x)}$.



5 If $f(x) = -2(x-2)^2 + 6$:

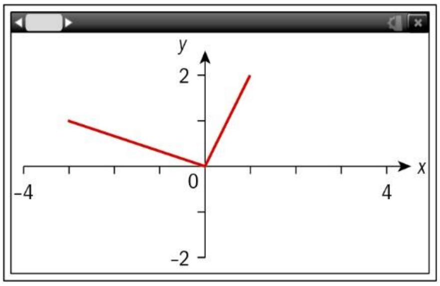
a Sketch the graph of $f(x)$.

b Sketch the graph of $f(x)$ reflected over the x-axis.

c **i** Sketch the graph of $f(x)$ reflected over the y-axis.

ii Write the equation of the function after the reflection over the y-axis.

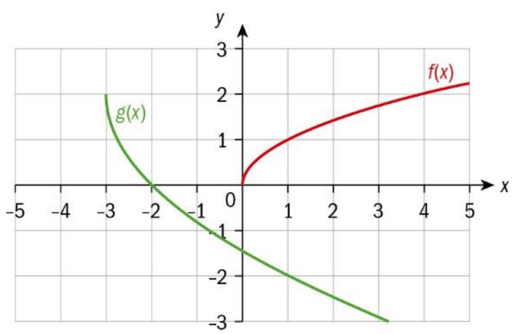
6 Given the graph of $y = f(x)$ below, sketch the graph of



a $y = -2f(x)$ **b** $y = f(x) + 3$ **c** $y = \frac{1}{f(x)} - 2$

7 The point $(-6, 8)$ is on the graph of $y = \frac{1}{2}f(3x) - 1$. State the coordinates of the corresponding point on the graph of $y = f(x)$.

8 The graphs of $g(x)$ and $f(x)$ are shown below.



a Identify the transformations performed on $f(x)$ to obtain $g(x)$.

b Write $g(x)$ in terms of $f(x)$.

9 Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{-3}{2x-4} - 1$. Describe the transformations needed to produce $g(x)$ from $f(x)$.

10 Describe the transformations needed to transform $f(x) = \frac{1}{x}$ to $g(x) = \frac{-3}{2x-4} - 1$.

11 Rewrite the equation of $y = \sqrt{x-2}$ after each of the following transformations.

a A translation of 2 units down and 3 units to the left.

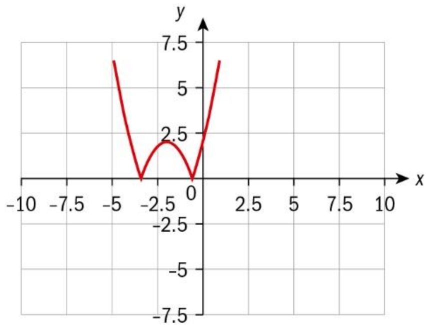
b A reflection in the y-axis and a vertical compression of $\frac{1}{2}$.

c A reflection in the x-axis and a horizontal compression by a factor of $\frac{1}{2}$.

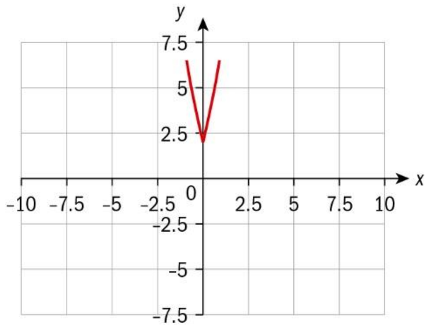
12 The point $(2, -4)$ lies on the graph of $y = -\frac{1}{2}x^3$. What are the coordinates of the corresponding point on the graph of $y + 1 = -\frac{1}{2}(2(x-3))^3 + 1$?

Answers

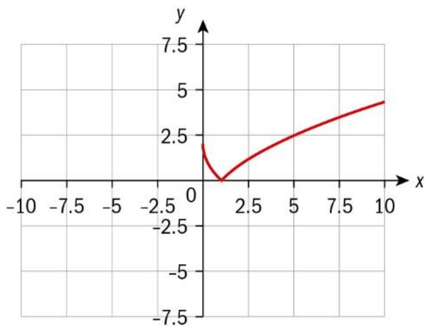
1 a $y = |f(x)|$:



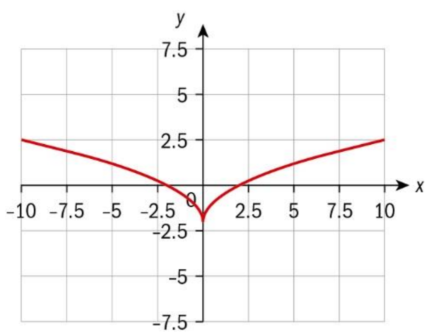
$y = f(|x|)$:



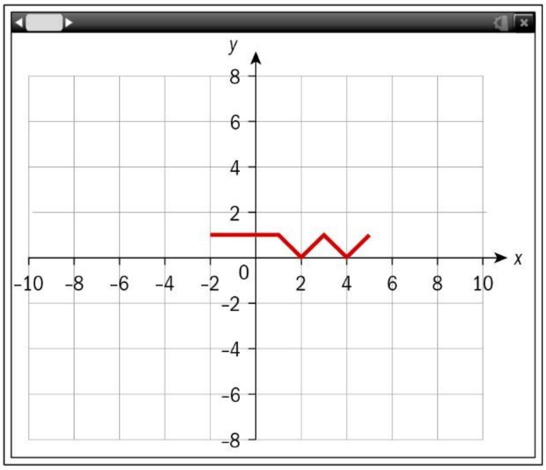
b $y = |f(x)|$:



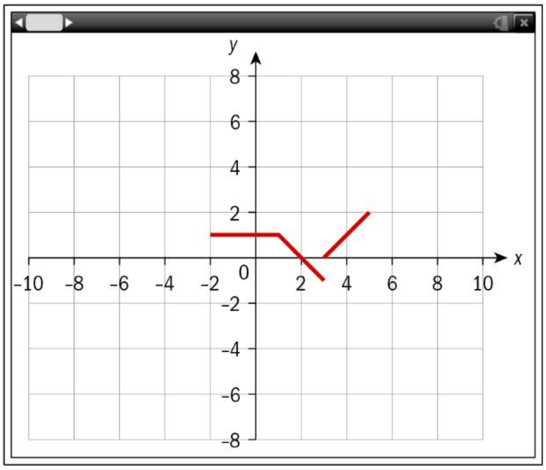
$y = f(|x|)$:



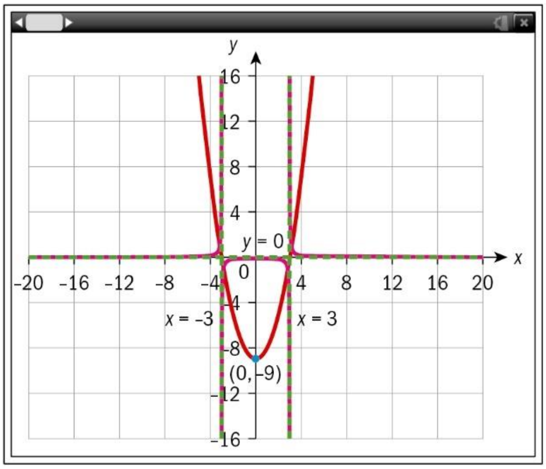
2 $y = |g(x)|$:


y
8
6
4
2
x
–10 –8 –6 –4 –2 0 2 4 6 8 10
–2
–4
–6
–8

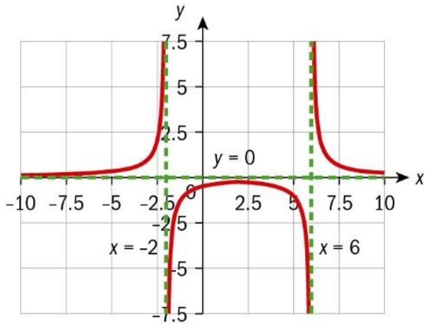
$y = f(|x|)$:


y
8
6
4
2
x
–10 –8 –6 –4 –2 0 2 4 6 8 10
–2
–4
–6
–8

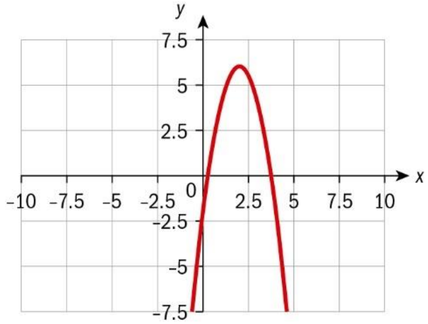
3


y
16
12
8
4
y = 0
x
–20 –16 –12 –8 –4 0 4 8 12 16 20
–4
x = –3
x = 3
–8
(0, –9)
–12
–16

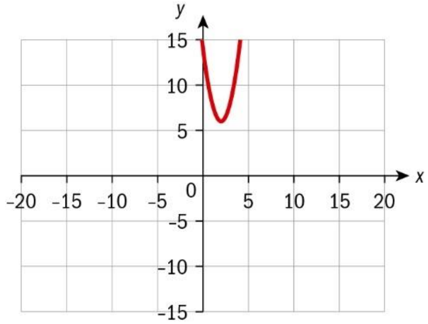
4



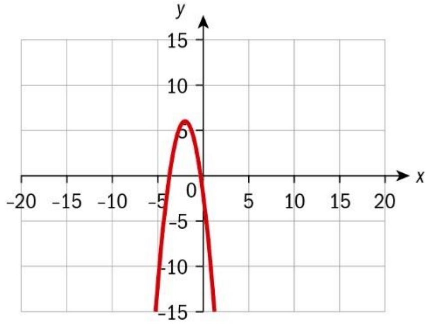
5 a



b

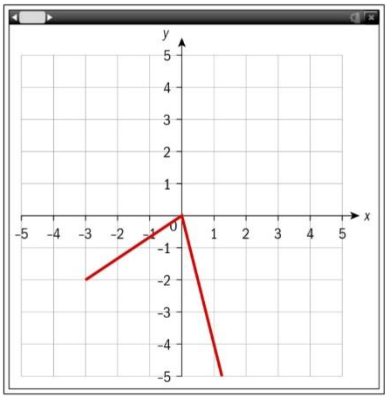


c i

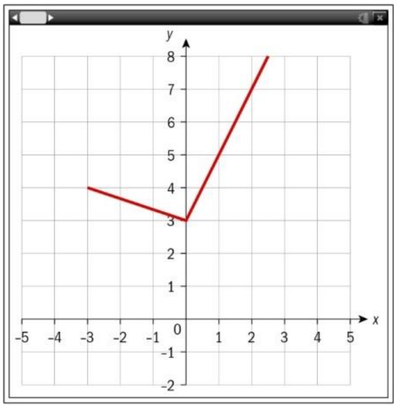


ii $y = -2(-(x+2))^2 + 6$

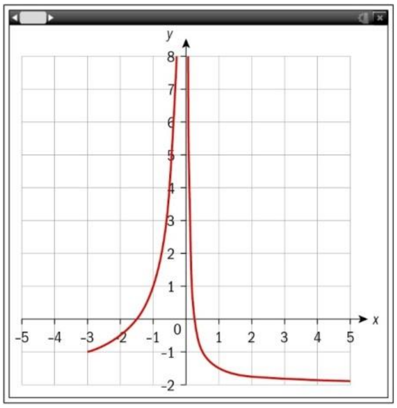
6 a



b



c



7 There was a vertical translation one unit down, so to reverse it, add 1 to the y-coordinate:

$(-6, 8) \to (-6, 9)$

There was a vertical compression by a factor of $\frac{1}{2}$, so to reverse it, multiply the y-coordinate by 2: $(-6, 9) \to (-6, 18)$

There was a horizontal compression by a factor of $\frac{1}{3}$, so to reverse it, multiply the x-coordinate by 3: $(-6, 18) \to (-18, 18)$

The coordinates of the point on $f(x)$ are $(-18, 18)$.

8 **a** Vertical reflection: $(x,y)\to(x,-y)$

Vertical stretch with a factor of 2: $(x,y)\to(x,2y)$

Horizontal translation three units to the left: $(x,y)\to(x+3,y)$

Vertical translation two units up: $(x,y)\to(x,y+2)$

b $g(x)=-2f(x+3)+2$

9 $g(x)=\frac{-3}{2x-4}-1=\frac{-3}{2(x-2)}-1$

Vertical reflection: $(x,y)\to(x,-y)$

Vertical stretch with a factor of 3: $(x,y)\to(x,3y)$

Horizontal compression with a factor of $\frac{1}{2}$: $(x,y)\to\left(\frac{1}{2}x,y\right)$

Horizontal translation two units to the right: $(x,y)\to(x-2,y)$

Vertical translation one unit down: $(x,y)\to(x,y-1)$

10 $g(x)=\frac{x-2}{x+4}=\frac{(x+4)-2-4}{x+4}=\frac{-6}{x+4}+1$

Vertical reflection: $(x,y)\to(x,-y)$

Vertical stretch with a factor of 6: $(x,y)\to(x,6y)$

Horizontal translation 4 units to the left: $(x,y)\to(x+4,y)$

Vertical translation one unit up: $(x,y)\to(x,y+1)$

11 **a** $y=\sqrt{x+1}-2$

b $y=\frac{1}{2}\sqrt{-(x-2)}$

c $y=-\sqrt{\frac{1}{2}(x-2)}$

12 Horizontal compression with a factor of $\frac{1}{2}$: $(x,y)\to\left(\frac{1}{2}x,y\right)$, $(2,-4)\to(1,-4)$

Horizontal translation 3 units to the right: $(x,y)\to(x-3,y)$, $(1,-4)\to(-2,-4)$

Vertical translation one unit up: $(x,y)\to(x,y+1)$, $(-2,-4)\to(-2,-3)$

The coordinates of the corresponding point are $(-2,-3)$.

3.1 Quadratic equations and inequalities

1 Solve the following quadratic equations by factorisation:

a $x^2+7x+10=0$

b $6x^2+17x-14=0$

2 Solve the following quadratic equations by completing the square, giving your answers in exact form:

a $x^2-5x-20=0$

b $4x^2+10x+3=0$

3 Solve the following quadratic equations by completing the square, giving your answers to 3 significant figures:

a $x^2+4x-3=0$

b $2x^2+7x+3=0$

4 Solve the following quadratic equations using the quadratic formula, giving your answer in exact form:

a $x^2+7x+3=0$

b $2x^2-5x=4$

5 Solve the equation $x^2+2kx=5k^2$ giving your answer in terms of k.

6 Without solving these quadratic equations, determine the nature of their roots:

a $x^2+2x-5=0$

b $6.25x^2+9=15x$

7 Find the values of m for which the equation $3x^2=2x+1-m$ has:

a two distinct real roots

b one real repeated root

c no real roots

8 Solve the following quadratic inequalities and verify using a GDC:

a $x^2 - 11x - 24 \le 0$

b $3x^2 > 2 - 5x$

Answers

1 a

$$(x+5)(x+2)=0$$
$$\Rightarrow x=-5 \text{ or } x=-2$$

b

$$6x^2+21x-4x-14=0$$
$$\Rightarrow 3x(2x+7)-2(2x+7)=0$$
$$\Rightarrow (3x-2)(2x+7)=0$$
$$\Rightarrow x=\frac{2}{3} \text{ or } x=-\frac{7}{2}$$

2 a

$$\left(x-\frac{5}{2}\right)^2=\frac{105}{4}$$
$$\Rightarrow x=\frac{5\pm\sqrt{105}}{2}$$

b

$$\left(x+\frac{5}{4}\right)^2=\frac{25}{16}-\frac{3}{4}$$
$$\Rightarrow \left(x+\frac{5}{4}\right)^2=\frac{13}{16}$$
$$\Rightarrow x=\frac{-5\pm\sqrt{13}}{4}$$

3 a

$$(x+2)^2=7$$
$$\Rightarrow x=-2\pm\sqrt{7}$$
$$\approx -4.65, 0.645$$

b

$$\left(x+\frac{7}{4}\right)^2=-2+\frac{49}{16}$$
$$\Rightarrow x=-\frac{7}{4}\pm\sqrt{\frac{17}{16}}$$
$$\approx -0.719, -3.81$$

4 a

$$x = \frac{-7 \pm \sqrt{37}}{2}$$

b

$$x = \frac{5 \pm \sqrt{57}}{4}$$

5

$$x = \frac{-2k \pm \sqrt{24k^2}}{2}$$
$$= k\left(-1 \pm \sqrt{6}\right)$$

6 a $\Delta = 4 + 20 > 0$

So 2 real roots

b $\Delta = (-15)^2 - 4 \times 6.25 \times 9 = 0$

So 1 repeated real root

7 $\Delta = 4 - 12(m-1) = 16 - 12m$

So

a 2 distinct roots when $m < \dfrac{4}{3}$

b 1 real repeated root when $m = \dfrac{4}{3}$

c no real roots when $m > \dfrac{4}{3}$

8 a

$$(x-8)(x-3) \le 0$$
$$\Rightarrow 3 \le x \le 8$$

b

$$3x^2 + 5x - 2 > 0$$
$$\Rightarrow (3x-1)(x+2) > 0$$
$$\Rightarrow x < -2, x > \frac{1}{3}$$

3.2 Complex numbers, modulus, operations with complex numbers

1 Find the real and imaginary parts of:

a $-3+6i$

b $\dfrac{8-15i}{17}$

2 Find the modulus and argument of the complex number in question 1.

3 Given $z_1 = 2-3i, z_2 = 6+i$ and $z_3 = -2+3i$ calculate

a $z_1 + z_2 - z_3$

b $\dfrac{2z_1 - 3z_2 + 4z_3}{5}$

4 For the complex numbers in question 3, calculate:

a $\dfrac{z_1}{z_2} - z_3$

b $\dfrac{2z_1 - 3z_3}{z_1 z_2^*}$

5 Find the real and imaginary parts of:

a $\dfrac{1+3i}{i}$

b $\dfrac{1+3i}{1-3i} - \dfrac{1-3i}{1+3i}$

6 Find the real numbers a and b that satisfy $(1+2i)(a+bi) = 1+7i$

7 Find the complex numbers z satisfying the following equations:

a $3(z+2i) = 2i(z-3)$

b $\dfrac{z+2+i}{1-3i} = \dfrac{z-2i+1}{2i+3}$

8 Solve for $z \in \mathbb{C} : |z| + z^* = 4+2i$

9 Prove that $\left(z_1 - z_2\right)^* = z_1^* - z_2^*$

10 Calculate:

a $i^5 + i^{11} + i^{17} + i^{23}$

b $\sum_{k=1}^{62} i^k$

11 Calculate:

a $\left(4+3i\right)^3$

b $\left(1-2i\right)^5$

12 Evaluate:

a $\sqrt{4i}$

b $\sqrt{8-15i}$

Answers

1 **a** Real part = -3, imaginary part = 6

b Real part = $\frac{8}{17}$, imaginary part = $-\frac{15}{17}$

2 **a** $\sqrt{45} = 3\sqrt{5}$

b $\frac{\sqrt{64+225}}{17} = 1$

3 **a** $(2-3i)+(6+i)-(-2+3i) = 10-5i$

b $\frac{4-6i-18-3i-8+12i}{5} = \frac{-22+3i}{5}$

4 **a**

$$\frac{2-3i}{6+i}+2+3i$$
$$=\frac{(2-3i)(6-i)}{37}+(-2+3i)$$
$$=\frac{9-20i}{37}-2+3i$$
$$=-\frac{65}{37}+\frac{91}{37}i$$

b

$$\frac{4-6i+6-9i}{(2-3i)(6-i)}$$
$$=\frac{10-15i}{9-20i}$$
$$=\frac{(10-15i)(9+20i)}{481}$$
$$=\frac{390}{481}+\frac{65}{481}i$$
$$=\frac{13}{37}+\frac{5}{37}i$$

5 **a**

$$\frac{1+3i}{i}=\frac{3-i}{1}$$

So real part = 3, imaginary part = -1

b

$$\frac{1+3i}{1-3i}-\frac{1-3i}{1+3i}$$
$$=\frac{(1+3i)^2-(1-3i)^2}{10}$$
$$=\frac{12i}{10}$$

So real part = 0, imaginary part = 1.2

6

$$a-2b=1, 2a+b=7$$
$$\Rightarrow a=3, b=1$$

7 a

$$3z+6i=2iz-6i$$
$$\Rightarrow z=\frac{-12i}{3-2i}$$
$$=\frac{-12i(3+2i)}{13}$$
$$=\frac{24}{13}-\frac{36}{13}i$$

b

$$\frac{z+2+i}{1-3i}=\frac{z-2i+1}{2i+3}$$
$$\Rightarrow (z+2+i)(2i+3)=(1-3i)(z-2i+1)$$
$$\Rightarrow 2iz+4i-2+3z+6+3i=z-2i+1-3iz-6-3i$$
$$\Rightarrow z(2i+3)+4+7i=z(1-3i)-5-5i$$
$$\Rightarrow z=\frac{-9-12i}{2+5i}=\frac{(-9-12i)(2-5i)}{29}$$
$$=-\frac{78}{29}+\frac{21}{29}i$$

8

$$\begin{aligned}
&|a+ib|+a-ib=4+2i\\
&\Rightarrow \sqrt{a^2+b^2}+a=4, b=-2\\
&\Rightarrow a^2+4=(4-a)^2\\
&\Rightarrow a=\frac{3}{2}, b=-2
\end{aligned}$$

9

$$\begin{aligned}
&(z_1-z_2)^*\\
&=((a_1-a_2)+(b_1-b_2i))^*\\
&=(a_1-a_2)-(b_1-b_2i)\\
&=(a_1-b_1i)-(a_2-b_2i)\\
&=z_1^*-z_2^*
\end{aligned}$$

10 a $i^5+i^{11}+i^{17}+i^{23}=i-i+i-i=0$

b $\sum_{k=1}^{62} i^k=(i-1-i+1)+(i-1-i+1)+\ldots..+(i-1-i+1)+i-1=i-1$

11 a

$$\begin{aligned}
&(4+3i)^3\\
&=64+144i-108-27i\\
&=-44+117i
\end{aligned}$$

b

$$\begin{aligned}
&(1-2i)^5\\
&=1-10i-40+80i+80-32i\\
&=41+38i
\end{aligned}$$

12 a

$$\begin{aligned}
&z^2=4i\Rightarrow(a+ib)^2=4i\\
&a^2-b^2=0, 2ab=4\\
&a^2-\frac{4}{a^2}=0\Rightarrow a=\pm\sqrt{2}\\
&\Rightarrow z=\sqrt{2}+\sqrt{2}i, -\sqrt{2}-\sqrt{2}i
\end{aligned}$$

b

$$a^2 - b^2 = 8, 2ab = -15$$

$$a^2 - \frac{225}{4a^2} = 8$$

$$\Rightarrow 4a^4 - 32a^2 - 225 = 0$$

$$\Rightarrow \left(2a^2 - 25\right)\left(2a^2 + 9\right) = 0$$

$$\Rightarrow a = \pm\frac{5\sqrt{2}}{2}$$

$$\Rightarrow z = \frac{5\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i, -\frac{5\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

3.3 Polynomials and their graphs, polynomial equations and inequalities

1 In each case use long division to find the quotient and remainder when f is divided by g.

a $f(x)=2x^3-3x^2+2x+11, g(x)=x-3$

b $f(x)=3x^4-7x^3+6x^2+2x+1, g(x)=x^2-x-2$

2 In each case use synthetic division to find the quotient and remainder when f is divided by g.

a $f(x)=x^3+3x^2-4x+1, g(x)=x-3$

b $f(x)=3x^5+10x^4-4x^3+15x^2-5x+23, g(x)=x+4$

3 $f(x)=2x^4+9x^3-12x^2-29x+30.$

a Show that $(x-1)$ and $(x+2)$ are factors.

b Hence fully factorise $f(x)$

4 In each case use long division to find the quotient and remainder when f is divided by g.

a $f(x)=4x^3-8x^2+3x+11, g(x)=2x+1$

b $f(x)=3x^5-7x^4+17x^3-8x^2+2x-4, g(x)=3x-1$

5 The polynomial $f(x)=x^4-2x^3+3x^2+ax+b$ is divisible by $(x+1)$ and $(x-2)$. Find a and b.

6 The polynomial $f(x)=4x^5+10x^4-3x^3+ax+b$ is divisible by (2x+1) and when it is divided by (x-2) the remainder is 20. Find a and b.

Answers

1 **a** Quotient $= 2x^2 - 5x + 7$, remainder $= -6$

b Quotient $= 3x^2 - 4x + 8$, remainder $= 2x + 17$

2 **a** Quotient $= x^2 + 6x + 14$, remainder $= 43$

b Quotient $= 3x^4 - 2x^3 + 4x^2 - x - 1$, remainder $= 27$

3

$$f(1) = 2 + 9 - 12 - 29 + 30 = 0$$
$$f(-2) = 32 - 72 - 48 + 58 + 30 = 0$$

So $(x-1)$ and $(x+2)$ are factors

Using long division

$$f(x) = (x-1)(x+2)(2x^2 + 7x - 15)$$
$$= (x-1)(x+2)(2x-3)(x+5)$$

4 **a** Quotient $= 2x^2 - 5x + 4$, remainder $= 7$

b Quotient $= x^4 - 2x^3 + 5x^2 - x - 1$, remainder $= -5$

5

$$1 + 2 + 3 - a + b = 0 \Rightarrow a - b = 6$$
$$16 - 16 + 12 + 2a + b = 0 \Rightarrow 2a + b = -12$$
$$\Rightarrow a = -2, b = -8$$

6

$$f\left(-\frac{1}{2}\right) = -\frac{1}{8} + \frac{5}{8} + \frac{3}{8} - \frac{1}{2}a + b = 0 \Rightarrow \frac{1}{2}a - b = \frac{7}{8}$$
$$f(2) = 128 + 160 - 24 + 2a + b = 20 \Rightarrow 2a + b = -244$$
$$\Rightarrow a = -\frac{389}{4}, b = -\frac{99}{2}$$

3.4 The fundamental theorem of algebra, sum and product of the zeros of polynomials

1 Find a polynomial of the smallest degree with integer coefficients whose zeros are:

a 0, 1 and -4

b $-1, -\frac{1}{3}, 1$ and 3

c $1, -2$ and $\sqrt[3]{2}$

2 Factorise these polynomials:

a $x^3 - 4x^2 - 7x + 10$

b $2x^4 - 3x^3 + 3x^2 + 5x - 3$

3 In each case, k is a zero with multiplicity n of the polynomial f. Factorise f fully.

a $k = 1, n = 2, f(x) = 2x^3 + x^2 - 8x + 5$

b $k = \frac{1}{2}, n = 3, f(x) = 8x^4 + 12x^3 - 30x^2 + 17x - 3$

4 In each case, f is a polynomial with complex zero z. Find the remaining zeros:

a $f(x) = x^3 - 2x^2 + 9x - 18, z = 3i$

b $f(x) = 4x^4 - 8x^3 + 19x^2 + 2x - 5, z = 1 + 2i$

5 Given that z is a zero of the polynomial f, find the missing real coefficients. Hence find all the remaining zeros.

a $z = -2, f(x) = x^3 + ax^2 - x + 2$

b $z = 1 + 2i, f(x) = x^4 + -5x^2 + ax + b$

6 Find the sum and product of the zeros of these polynomials:

a $f(x) = 3x^5 + 2x^4 - 7x^3 + 2x^2 - x - 9$

b $f(x) = 4x^8 - 5x^7 + 3x^6 - 2x^5 + 9x^4 - 7x + 6$

7 The quartic equation $4x^4 - 2x^3 + 3x^2 - 5x - 3 = 0$ has roots x_1, x_2, x_3 and x_4. Find the value of

$$\frac{1}{x_1x_2x_3} + \frac{1}{x_1x_2x_4} + \frac{1}{x_1x_3x_4} + \frac{1}{x_2x_3x_4}.$$

Answers

1 a $x(x-1)(x+4)=x^3+3x^2-4x$

b

$$
\begin{aligned}
&(x+1)(3x+1)(x-1)(x-3)\\
&=(x+1)(3x+1)(x^2-4x+3)\\
&=(x+1)(3x^3-11x^2+4x+3)\\
&=3x^4-8x^3-6x^2+8x+3
\end{aligned}
$$

c

$$
\begin{aligned}
&(x-1)(x+2)(x^3-2)\\
&=(x^2+x-2)(x^3-2)\\
&=x^5+x^4-2x^3-2x^2-2x-4
\end{aligned}
$$

2 a

$$
\begin{aligned}
&(x-1)(x^2-3x-10)\\
&=(x-1)(x-5)(x+2)
\end{aligned}
$$

b

$$
\begin{aligned}
&(x+1)(2x^3-5x^2+8x-3)\\
&=(x+1)(2x-1)(x^2-2x+3)
\end{aligned}
$$

3 a

$$
\begin{aligned}
&(x^2-2x+1)(2x+5)\\
&=(x^2-1)^2(2x+5)
\end{aligned}
$$

b

$$
\begin{aligned}
&(8x^3-12x^2+6x-1)(x+3)\\
&=(2x-1)^3(x+3)
\end{aligned}
$$

4 a $z=-3i$ is also a zero $\Rightarrow x^2+9$ is a factor

$\Rightarrow f(x)=(x^2+9)(x-2)$

So other zeros are 2 and $-3i$.

b $z=1-2i$ is also a zero $\Rightarrow x^2-2x+5$ is a factor

$$\Rightarrow f(x)=\left(x^2-2x+5\right)\left(4x^2-1\right)$$

So other zeros are $1-2i,\pm\frac{1}{2}.$

5 **a**

$$-8+4a+2+2=0\Rightarrow a=1$$
$$\Rightarrow f(x)=(x+2)\left(x^2-x+1\right)$$

So no other zeros since $\Delta<0.$

b $-1-i$ is also a zero

$\Rightarrow x^2+2x+2$ is a factor

So by equating coefficients

$$f(x)=\left(x^2+2x+2\right)\left(x^2-2x-1\right)$$

So $a=-6, b=-2$

So remaining zeros are $-1-i, 1\pm\sqrt{3}.$

6 **a** Sum = $-\frac{2}{3}$, product = $\frac{9}{3}=3.$

b Sum = $\frac{5}{4}$, product = $\frac{6}{4}=\frac{3}{2}.$

7

$$\frac{1}{x_1x_2x_3}+\frac{1}{x_1x_2x_4}+\frac{1}{x_1x_3x_4}+\frac{1}{x_2x_3x_4}$$
$$=\frac{x_1+x_2+x_3+x_4}{x_1x_2x_3x_4}$$
$$=\frac{-\frac{2}{4}}{-\frac{3}{4}}$$
$$=\frac{2}{3}.$$

3.5 Solving equations and inequalities

1 Solve the following equations:

a $x^3 - 2x^2 - 5x + 6 = 0$

b $x^4 - 3x^3 - 3x^2 + 7x + 6 = 0$

2 The equation $2x^3 + ax^2 + bx - 12 = 0$ has a repeated root of -2.

a Find the values of a and b.

b Find the remaining root.

3 Solve the following polynomial inequalities:

a $x^3 - 2x^2 - 5x + 6 \leq 0$

b $4x^4 + 4x^3 - 3x^2 - 4x + 1 > 0$

4 Use a calculator to solve the inequality $x^7 + x^6 > 3x - 4$.

Answers

1 a

$$(x-1)(x^2-x-6)=0$$
$$\Rightarrow (x-1)(x-3)(x+2)=0$$
$$\Rightarrow x=-2,1,3.$$

b

$$(x+1)^2(x^3-4x^2+x+6)=0$$
$$\Rightarrow (x+1)^2(x^2-5x+6)=0$$
$$\Rightarrow (x+1)^2(x-3)(x-2)=0$$
$$\Rightarrow x=-1,2,3.$$

2 $2x^3+ax^2+bx-12=(x+2)^2(cx+d)$

By equating coefficients, $c=2, d=-3$

So polynomial is $(x+2)^2(2x-3)$

So $a=5, b=-4$ and remaining root is $\frac{3}{2}$.

3 a

$$(x-1)(x^2-x-6)\le 0$$
$$\Rightarrow (x-1)(x-3)(x+2)\le 0$$
$$\Rightarrow x\le 2, 1\le x\le 3.$$

b

$$(x-1)(4x^3+8x^2+5x+1)>0$$
$$\Rightarrow (x-1)(x+1)(4x^2+4x+1)>0$$
$$\Rightarrow (x-1)(x+1)(2x+1)^2>0$$
$$\Rightarrow x<-1, x>1.$$

4 $x^7+x^6-3x+4>0$

Solving $x^7+x^6-3x+4=0$ on a calculator gives

$x=-1.573\ldots\approx -1.57$

And by also looking at the graph

$x > -1.57$

3.6 Solving systems of linear equations

1 Find the value of the real parameter m such that the system of equations

$$\begin{cases} 2x + my = -1 \\ x + 3y = 5 \end{cases}$$

has no solution.

2 Find the value of the real parameter p such that the system of equations

$$\begin{cases} 3x - y = 1 \\ 3px - 2y = p \end{cases}$$

has infinitely many solutions.

3 Find the values of the real parameter s such that the system of equations

$$\begin{cases} 3x - sy = 2 \\ x + y = 5 \end{cases}$$

has a unique solution.

Hence find the solution in terms of s.

4 Solve the system of equations

$$\begin{cases} (1-i)x + 2y = 4 - 4i \\ 4x - (1+i)y = 1 + 5i \end{cases}$$

5 **a** Find the value of the real parameter k such that the system of equations

$$\begin{cases} 3x + y + z = 1 \\ x - y + 2z = 3 \\ 2x - 6y + kz = p \end{cases}$$

does not have a unique solution.

b In the case where the system has an infinite number of solutions, find the value of p and the solutions.

6 Find the quadratic function f(x) whose graph passes through the points (1,3), (-1,10) and (2,7).

Answers

1 $m = 6.$

2 $p = 2.$

3

$s \neq -3$

$x = \dfrac{2+5s}{3+s}, y = \dfrac{13}{3+s}$

4 $x = 1+i, y = 1-2i.$

5 Adding first two equations

$4x + 3z = 4$

Eliminating y from equations 1 and 3

$20x + (6+k)z = 6 + p$

so no unique solution if

and infinite number of solutions if $k = 9$

In this case

$z = \dfrac{4-4x}{3}$

And substituting this in equation 1

$y = 1 - \dfrac{4-4x}{3} - 3x = \dfrac{-1-5x}{3}$

So solutions are of the form $\left(x, \dfrac{-1-5x}{3}, \dfrac{4-4x}{3}\right)$

6 Writing $f(x) = ax^2 + bx + c,$

$a + b + c = 3$

$a - b + c = 10$

$4a + 2b + c = 7$

Solving these gives

$a = 2, b = -3, c = 5$

So $f(x) = 2x^2 - 3x + 5.$

4.1 Limits, continuity and convergence

1 Find the limit of each function, if it exists.

a $\lim_{x \to -4} \frac{x^2 - 16}{x + 4}$

b $\lim_{x \to 2} \begin{cases} x & x < 2 \\ 2x + 1 & x>2 \end{cases}$

c $\lim_{x \to 1} \begin{cases} x - 1 & x < 1 \\ x^2 - 1 & x>1 \end{cases}$..

2 **a** Determine whether $f(x) = \begin{cases} 2x - 1 & x < 2 \\ 3x - 2 & x>2 \end{cases}$ is continuous at x=1.

b Determine whether $f(x) = \begin{cases} \frac{(x-2)^2}{x-2} & x \neq 2 \\ 1 & x = 2 \end{cases}$ is continuous at x=2.

c Find the value of k such that $f(x) = \begin{cases} kx^2 & x > -2 \\ 8 & x < -2 \end{cases}$ is continuous at x=-2.

3 Determine whether each function is continuous on the set of real numbers. If the function is not continuous, state the values of x for which the function is discontinuous.

a $f(x) = \frac{x^2 + 4}{x^2 - 4}$

b $f(x) = \frac{x}{x^2 + 1}$

c $f(x) = \frac{x^3}{x^3 - 8}$

4 **Do not use a calculator for this question.**

Find the required limit, if it exists.

a $\lim_{x \to 4} \frac{x + 1}{x - 1}$

b $\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$

c $\lim_{x \to 1} \frac{4x^2 - 2(1 + x)}{x - 1}$

5 **Do not use a calculator for this question.**

Find $\lim_{x \to \infty} f(x)$ if it exists.

a $f(x) = \dfrac{2x}{x+1}$

b $f(x) = \dfrac{2x^2 - 1}{x+1}$

c $f(x) = \dfrac{x+1}{x^2 - x - 2}$

6 **Do not use a calculator for this question.**

Find any vertical and horizontal asymptotes of these functions.

a $f(x) = \dfrac{x}{x+1}$

b $f(x) = \dfrac{2x}{x^2 - 3}$

c $f(x) = \dfrac{x^3 - 2x}{x^2 - 1}$

7 **Do not use a calculator for this question.**

Determine whether these sequences converge. If the sequence converges, state the number it converges to.

a 1, 2, 1, 2, 1, 2, …

b $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots\ldots$

c $u_n = \dfrac{n+1}{2n+1}$

d $u_n = \dfrac{n+1}{n^2+1}$

e $u_n = \dfrac{n^3+1}{n-1}$

8 **Do not use a calculator for this question.**

Determine whether each series converges. If it converges, determine its sum.

a $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

b $\displaystyle\sum_{n=0}^{\infty} \frac{5^n - 2^n}{7^n}$

c $\displaystyle\sum_{n=1}^{\infty} \left(\frac{e}{2.7}\right)^n$

9 **Do not use a calculator for this question.**

Find the set of values for which $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2x}{x+3}\right)^n$ converges.

Answers

1 From GDC

a -8

b No limit

c 0

2 **a** Yes, because both limits = 1.

b No because the left limit = 0 and the right limit =1

c $k(-2)^2 = 8 \Rightarrow k = 2$

3 **a** The function is discontinuous at x = ± 2 because the denominator is zero and the numerator is not.

b Continuous for all real values.

c The function is discontinuous at x = 2 because the denominator is zero and the numerator is not.

4 **a** $\frac{5}{3}$

b $\frac{x^2-1}{x^2-x} = \frac{(x-1)(x+1)}{x(x-1)} = \frac{x+1}{x}$ so limit =2.

c $\frac{4x^2-2(1+x)}{x-1} = \frac{2(2x+1)(x-1)}{x-1} = 2(2x+1)$ so limit = 6.

5 **a** $\frac{2x}{x+1} = 2 - \frac{2}{x+1}$ so limit =2.

b $\frac{2x^2-x}{x+1} = 2x + \frac{-3x}{x+1}$ so no finite limit exists

c $\frac{x+1}{x^2-x-2} = \frac{1}{x-2}$ so limit = 0.

6 **a** $\frac{x}{x+1} = 1 - \frac{1}{x+1}$ so vertical asymptote of x=-1 and horizontal asymptote of y = 1.

b $\frac{2x}{x^2-3} \to 0$ as $x \to \infty$

so vertical asymptotes of $x = \pm\sqrt{3}$ and horizontal asymptote of $y = 0$.

c $\frac{x^3-2x}{x^2-1} = x + \frac{-x}{x^2-1}$ so vertical asymptotes at $x = \pm 1$. No horizontal asymptotes.

7 **a** Oscillating.

b Limit = 0.

c $\lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2}$

d $\lim_{n \to \infty} \frac{n+1}{n^2+1} = \lim_{n \to \infty} \frac{1+\frac{1}{n}}{n+\frac{1}{n}} = 0$

e $\lim_{n \to \infty} \frac{n^3+1}{n-1} = \lim_{n \to \infty} \frac{n^2+\frac{1}{n}}{1-\frac{1}{n}}$ so diverges.

8 **a** Geometric with $|r| < 1$ so converges to $\frac{1}{1-\frac{1}{2}} = 2$.

b $\sum_{n=0}^{\infty} \frac{5^n - 2^n}{7^n} = \sum_{n=0}^{\infty} \frac{5^n}{7^n} - \frac{2^n}{7^n}$ so converges to $\frac{1}{1-\frac{5}{7}} - \frac{1}{1-\frac{2}{7}} = \frac{7}{2} - \frac{7}{5} = 2.1$

c $\frac{e}{2.7} > 1$ so diverges.

9 $\left|\frac{2x}{x+3}\right| < 1 \Rightarrow -1 < x < 3.$

4.2 The derivative of a function

1 Find the gradient of the graph of the function at the given value of x.

a $y = -\dfrac{3}{x} : x = 1$

b $y = x^2 + 3x - 2 : x = 0$

c $y = \dfrac{2x}{x-1} : x = 2$

2 Find the gradient function of $f(x) = x^2 + x - 3$.

3 Find the coordinates of the point on the graph of $y = x^2 + 2$ where the gradient is 4.

4 Differentiate the following functions from first principles.

a $f(x) = 2x^2 + x - 3$

b $f(x) = \sqrt{x-3}$

5 A particle's displacement is $10t - t^3$.

a Find an expression for the velocity.

b Find the velocity after 2 seconds and comment on what the sign means.

6 Find the point on the curve $y = \dfrac{1}{x+1}$ where the derivative is -1 and find the equation of the tangents at this point.

7 Find any points on the curve $y = x^3 - 12x$ where the tangent is horizontal.

Answers

1 **a** Gradient $= \lim_{h \to 0} \dfrac{-\dfrac{3}{1+h} - -3}{h}$

$= \lim_{h \to 0} \dfrac{\dfrac{-3+3+3h}{1+h}}{h}$

$= \lim_{h \to 0} \dfrac{3}{1+h} = 3$

b Gradient $= \lim_{h \to 0} \dfrac{\left(h^2+3h-2\right)-(-2)}{h}$

$= \lim_{h \to 0} \dfrac{h^2+3h}{h}$

$= \lim_{h \to 0} (h+3) = 3$

c Gradient $= \lim_{h \to 0} \dfrac{\dfrac{4+2h}{2+h-1} - 4}{h}$

$= \lim_{h \to 0} \dfrac{\dfrac{4+2h-4-4h}{h+1}}{h}$

$= \lim_{h \to 0} \dfrac{-2}{h+1} = -2$

2 $\lim_{h \to 0} \dfrac{(x+h)^2+(x+h)-3-\left(x^2+x-3\right)}{h}$

$= \lim_{h \to 0} \dfrac{2xh+h^2+h}{h}$

$= \lim_{h \to 0} 2x+h+1$

$= 2x+1$

3 Gradient function is given by

$\lim_{h \to 0} \dfrac{(x+h)^2+2-\left(x^2+2\right)}{h}$

$= \lim_{h \to 0} \dfrac{2xh+h^2}{h}$

$= \lim_{h \to 0} 2x+h$

$= 2x$

$\Rightarrow x = 2$

and $y = 6$

So coordinates are (2,6)

4 **a** $f'(x) = \lim_{h \to 0} \dfrac{2(x+h)^2 + x + h - 3 - (2x^2 + x - 3)}{h}$

$= \lim_{h \to 0} \dfrac{4xh + 2h^2 + h}{h}$

$= \lim_{h \to 0} 4x + 2h + 1$

$= 4x + 1$

B $f'(x) = \lim_{h \to 0} \dfrac{\sqrt{x+h-3} - \sqrt{x-3}}{h}$

$= \lim_{h \to 0} \dfrac{\left(\sqrt{x+h-3} - \sqrt{x-3}\right)\left(\sqrt{x+h-3} + \sqrt{x-3}\right)}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)}$

$= \lim_{h \to 0} \dfrac{h}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)}$

$= \lim_{h \to 0} \dfrac{1}{\left(\sqrt{x+h-3} + \sqrt{x-3}\right)}$

$= \dfrac{1}{2\sqrt{x-3}}$

5 **a** $\lim_{h \to 0} \dfrac{10(t+h) - (t+h)^3 - (10t - t^3)}{h}$

$= \lim_{h \to 0} \dfrac{10h - 3t^2h - 3th^2 - h^3}{h}$

$= \lim_{h \to 0} 10 - 3t^2 - 3th - h^2$

$= 10 - 3t^2$

b Velocity =10-12=-2

Negative sign means it is travelling backwards.

6 Derivative = $\lim_{h \to 0} \dfrac{\dfrac{1}{x+h+1} - \dfrac{1}{x+1}}{h}$

$= \lim_{h \to 0} \dfrac{\dfrac{-h}{(x+1)(x+h+1)}}{h}$

$= \lim_{h \to 0} \dfrac{-1}{(x+1)(x+h+1)}$

$= -\dfrac{1}{(x+1)^2}$

So derivative =-1 when x=0

And y=1 so equation is $y = -x + 2$

7 Derivative = $\lim_{h \to 0} \frac{(x+h)^3 - 12(x+h) - (x^3 - 12x)}{h}$

$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 12h}{h}$

$= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 12$

$= 3x^2 - 12$

$3x^2 - 12 = 0$ when $x = \pm 2$

So coordinates are (-2, 16), (2, -16)

4.3 Differentiation Rules

1 Find $\frac{dy}{dx}$ for each function.

a $y = (2x+1)^2$

b $y = \frac{1}{2}x^3 - 3x + 1$

c $y = \sqrt[3]{x}$

d $y = \sqrt[4]{x^3}$

e $y = \left(1-\sqrt{x}\right)\left(2+\sqrt[3]{x}\right)$

2 Find the equation of the tangent and normal to the curve $y = 3x^2 - 5x$ at $x = 1$.

3 Find the coordinates of the point where the normal to the curve $y = x^2 - 4x + 2$ at $x = 0$ intersects the curve again.

4 Find $\frac{dy}{dx}$ for each function.

a $y = (2x-1)^6$

b $y = -\frac{3}{\sqrt{1+x^2}}$

c $y = \frac{1}{\sqrt[3]{x^2-3}}$

5 Find the equation of the tangent and normal to the curve $y = \sqrt{x^2+1}$ at $x = 1$.

6 Differentiate these functions with respect to x.

a $y = (2x-1)(x+5)^3$

b $y = x\sqrt{2x-3}$

c $y = \frac{1}{x^3+3x^2-1}$

7 Consider the function $y = \sqrt{x^2+1}\,(x+1)^2$.

a Show that $\frac{dy}{dx} = \frac{(x+1)\left(3x^2+x+2\right)}{\sqrt{x^2+1}}$.

b Find the coordinates of any points on the curve where the tangent is horizontal.

8 Differentiate using the quotient rule.

a $y = \dfrac{1+x}{3x-2}$

b $y = \dfrac{2x-1}{x^2+2x}$

9 Differentiate the functions with respect to x.

a $y = \dfrac{x^2+1}{x^3-1}$

b $y = \dfrac{1+2\sqrt{x}}{1-\sqrt{x}}$

10 $f(x) = 3x^4 - 7x^3 + 2x^2 - 5x + 7$. Find $f'''(2)$.

11 Find the coordinates of the points on the curve $y = x^4 - 6x^3 + 12x^2 - 3x - 1$ where $\dfrac{d^2y}{dx^2} = 0$.

Answers

1 **a** $y = 4x^2 + 4x + 1$

$\Rightarrow \dfrac{dy}{dx} = 8x + 4$

b $\dfrac{dy}{dx} = \dfrac{3}{2}x^2 - 3.$

c $y = x^{\frac{1}{3}} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}}$

d $y = x^{\frac{3}{4}} \Rightarrow \dfrac{dy}{dx} = \dfrac{3}{4}x^{-\frac{1}{4}} = \dfrac{3}{4\sqrt[4]{x}}$

e $y = 2 + x^{\frac{1}{3}} - 2x^{\frac{1}{2}} - x^{\frac{5}{6}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}} - x^{-\frac{1}{2}} - \dfrac{5}{6}x^{-\frac{1}{6}}$

$= \dfrac{1}{3\sqrt[3]{x^2}} - \dfrac{1}{\sqrt{x}} - \dfrac{5}{6\sqrt[6]{x}}$

2 $\dfrac{dy}{dx} = 6x - 5 = 1$ at $x = 1$

$y = -2$

Equation of tangent is $y = x - 3$

Equation of normal is $y = -x - 1$

3 $y = 2$ when $x = 0$

$\dfrac{dy}{dx} = 2x - 4$

So gradient of normal = $\dfrac{1}{4}$

Equation of normal is $y = \dfrac{1}{4}x + 2$

Intersect when $\dfrac{1}{4}x + 2 = x^2 - 4x + 2$

$\Rightarrow x^2 - 4\dfrac{1}{4}x = 0$

$\Rightarrow$ Second point of intersection is $\left(\dfrac{17}{4}, \dfrac{49}{16}\right)$

4 **a** $\dfrac{dy}{dx} = 6(2x-1)^5 \times 2 = 12(2x-1)^5$

b $y = -3(1+x^2)^{-\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3}{2}(1+x^2)^{-\frac{3}{2}} \times 2x = \dfrac{6x}{2\sqrt{(1+x^2)^3}}$

c $y = (x^2 - 3)^{-\frac{1}{3}}$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{3}\left(x^2-3\right)^{-\frac{4}{3}} \times 2x = \frac{2x}{3\sqrt[3]{\left(1+x^2\right)^4}}$$

5 $y = \left(x^2+1\right)^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \times 2x = \frac{1}{\sqrt{2}} \text{ at } x = 1$$

$y = \sqrt{2}$ when $x = 1$

Equation of tangent is $y = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}$

Equation of normal is $y = -\sqrt{2}x + 2\sqrt{2}$

6 **a** $\frac{dy}{dx} = 2\left(x+5\right)^3 + \left(2x-1\right) \times 3\left(x+5\right)^2$

$$= \left(x+5\right)^2\left(2\left(x+5\right)+3\left(2x-1\right)\right)$$
$$= \left(x+5\right)^2\left(8x+7\right)$$

b $y = x\left(2x-3\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \left(2x-3\right)^{\frac{1}{2}} + x \times \frac{1}{2}\left(2x-3\right)^{-\frac{1}{2}} \times 2$$

$$= \sqrt{2x-3} + \frac{x}{\sqrt{2x-3}}$$
$$= \frac{3x-3}{\sqrt{2x-3}}$$

c $\frac{dy}{dx} = -\frac{3x^2+6x}{\left(x^3+3x^2-1\right)^2}$

7 **a** $y = \left(x^2+1\right)^{\frac{1}{2}}\left(x+1\right)^2$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \times 2x \times \left(x+1\right)^2 + \left(x^2+1\right)^{\frac{1}{2}} \times 2\left(x+1\right)$$

$$= \frac{x\left(x+1\right)^2 + 2\left(x^2+1\right)\left(x+1\right)}{2\left(x-1\right)^{\frac{1}{2}}}$$
$$= \frac{\left(x+1\right)\left(x^2+x+2x^2+2\right)}{2\sqrt{x-1}}$$
$$= \frac{\left(x+1\right)\left(3x^2+x+2\right)}{\sqrt{x^2+1}}$$

b $\frac{dy}{dx} = 0$ only when $x = -1$ since $3x^2 + x + 2$ never = 0.

So tangent = 0 at $\left(-1, 4\sqrt{2}\right)$

8 **a** $\frac{dy}{dx} = \frac{1 \times (3x-2) - (1+x) \times 3}{(3x-2)^2}$

$= -\frac{5}{(3x-2)^2}$

B $\frac{dy}{dx} = \frac{2 \times (x^2+2x) - (2x-1) \times (2x+2)}{(x^2+2x)^2}$

$= \frac{2x^2 + 4x - 4x^2 - 2x + 2}{(x^2+2x)^2}$

$= \frac{-2x^2 + 2x + 2}{(x^2+2x)^2}$

9 **a** $\frac{dy}{dx} = \frac{2x \times (x^3-1) - (x^2+1) \times 3x^2}{(x^3-1)^2}$

$= \frac{2x^4 - 2x - 3x^4 + 3x^2}{(x^3-1)^2}$

$= \frac{-x^4 + 3x^2 - 2x}{(x^2+2x)^2}$

b $\frac{dy}{dx} = \frac{x^{-\frac{1}{2}} \times \left(1 - x^{\frac{1}{2}}\right) - \left(1 + 2x^{\frac{1}{2}}\right) \times -\frac{1}{2}x^{-\frac{1}{2}}}{\left(1 - x^{\frac{1}{2}}\right)^2}$

$= \frac{x^{-\frac{1}{2}} - 1 - \frac{1}{2}x^{-\frac{1}{2}} - 1}{\left(1 - x^{\frac{1}{2}}\right)^2}$

$= \frac{\frac{1}{2}x^{-\frac{1}{2}} - 2}{\left(1 - x^{\frac{1}{2}}\right)^2}$

$= \frac{1 - 4\sqrt{x}}{2\sqrt{x}\left(1 - \sqrt{x}\right)^2}$

10 $f'(x) = 12x^3 - 21x^2 - 4x - 5$

$f''(x) = 36x^2 - 42x - 4$

$f'''(x) = 72x - 42$

So $f'''(2) = 102$

11 $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x - 3$

$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24$

So $\frac{d^2y}{dx^2} = 0$ when $12x^2 - 36x + 24 = 0$

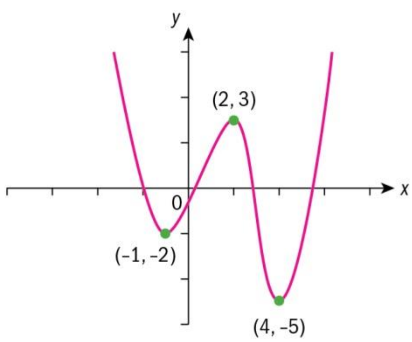
$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow x = 1, 2$

$\Rightarrow$ coordinates are $(1,3), (2,9)$

4.4 Graphical interpretation of the derivatives

1 The graph of y=f(x) is shown below.



State the values of x where

a $f' = 0$

b f' is positive

c f' is negative

d f is increasing

e f is decreasing

2 For each function

a Find any stationary points and determine the nature of these points.

b State the intervals on which the function is increasing.

c State the intervals on which the function is decreasing.

i $y = x^3 - 3x^2 + 1$

ii $y = 2x + \dfrac{1}{x+1}$

3 The graph of the function $y = x^3 + ax^2 + bx + c$ passes through the point (2, -39) and has turning points at x = -1, 5. Find the values of a, b and c.

4 **Do not use a GDC for this question.**

Find and classify any turning points of the function $y = 3x^3 - 4x^2$

5 **Do not use a calculator for this question.**

The function $f(x) = x^3 + ax^2 + b$ has a turning point at (-2,8).

a Find a and b.

b Find any other turning points

6 **Do not use a calculator for this question.**

For these functions, find

a Any points of inflexion.

b The intervals where the function is concave up.

c The intervals where the function is concave down.

i $y = x^4 - x^2$

ii $y = x^3 + x^2 - 3$

iii $y = 3x^4 + 2x^3 - 7$

7 The cubic function $f(x) = ax^3 + bx^2 + cx + d$ has no stationary points. Show that $b^2 < 3ac$.

Answers

1 **a** $x=-1, 2, 4$

b b. $-1<x<2$, $x>4$

c c. $x<-1$, $2<x<4$

d d. $-1<x<2$, $x>4$

e e. $x<-1$, $2<x<4$

2 **a** **i**

$\frac{dy}{dx} = 3x^2 - 6x = 0$ when $x = 0, 2$

So stationary points at (0,1), (2,-3)

(0,1) is a maximum and (2,-3) is a minimum.

ii Increasing when $x<0$, $x>2$

iii Decreasing when $0<x<2$.

b **i**

$\frac{dy}{dx} = 2 - \frac{1}{(x+1)^2} = 0$ when $x = -1 \pm \frac{1}{\sqrt{2}}$

So stationary points at $\left(-1-\frac{1}{\sqrt{2}}, -2-2\sqrt{2}-\frac{1}{\sqrt{2}}\right), \left(-1+\frac{1}{\sqrt{2}}, -2+2\sqrt{2}+\frac{1}{\sqrt{2}}\right)$

$\left(-1-\frac{1}{\sqrt{2}}, -2-2\sqrt{2}-\frac{1}{\sqrt{2}}\right)$ is a maximum

and $\left(-1+\frac{1}{\sqrt{2}}, -2+2\sqrt{2}+\frac{1}{\sqrt{2}}\right)$ is a minimum.

ii Increasing when $x < -1-\frac{1}{\sqrt{2}}, x > -1+\frac{1}{\sqrt{2}}$

iii Decreasing when $-1-\frac{1}{\sqrt{2}} < x < -1+\frac{1}{\sqrt{2}}$

3 $\frac{dy}{dx} = 3x^2 + 2ax + b$

$\Rightarrow 3 - 2a + b = 0, 75 + 10a + b = 0$

$\Rightarrow a = -6, b = -15$

$8 + 4a + 2b + c = -39 \Rightarrow c = 7$

4 $\frac{dy}{dx} = 9x^2 - 8x = 0$ when $x = 0, \frac{8}{9}$.

So stationary points at $(0,0), \left(\frac{8}{9}, -\frac{256}{243}\right)$

and by looking at sign of derivative or y values on either side

$(0,0)$ is a maximum and $\left(\frac{8}{9}, -\frac{256}{243}\right)$ is a minimum.

5 **a** $\frac{dy}{dx} = 3x^2 + 2ax \Rightarrow 12 - 4a = 0 \Rightarrow a = 3$

$-8 + 4a + b = 8 \Rightarrow b = 4$

b Other turning point at $(0, 4)$

6 **a** **I** $\frac{d^2y}{dx^2} = 12x^2 - 2$

Points of inflexion at $x = \pm\frac{1}{\sqrt{6}}$

$\left(\frac{1}{\sqrt{6}}, -\frac{5}{36}\right), \left(-\frac{1}{\sqrt{6}}, -\frac{5}{36}\right)$

ii Concave up when $x < -\frac{1}{\sqrt{6}}, x > \frac{1}{\sqrt{6}}$

iii Concave down when $-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$

b **i** $\frac{d^2y}{dx^2} = 6x + 2$

Point of inflexion at $x = -\frac{1}{3}$

$\left(-\frac{1}{3}, -\frac{79}{27}\right)$

ii Concave up when $x > -\frac{1}{3}$

iii Concave down when $x < -\frac{1}{3}$

c **I** $\frac{d^2y}{dx^2} = 36x^2 - 12x$

Points of inflexion at $x = 0, \frac{1}{3}$

$(0, -7), \left(\frac{1}{3}, -\frac{62}{9}\right)$

ii Concave up when $x < 0, x > \frac{1}{3}$

iii Concave down when $0 < x < \frac{1}{3}$

7 $\frac{dy}{dx} = 3ax^2 + 2bx + c$

No stationary points $\Rightarrow$ discriminant of $\frac{dy}{dx} < 0$

$\Rightarrow (2b)^2 - 12ac < 0 \Rightarrow b^2 < 3ac$

4.5 Applications of differential calculus

1 **Do not use a calculator for this question.**

A farmer has 160m of fencing and wishes to build a rectangular plot of land, one side of which will be bounded by an existing wall. What dimensions will enable her to enclose the maximum area?

2 **Do not use a calculator for this question.**

A closed cylinder has a surface area of $10\pi \text{ cm}^2$. What is the maximum possible volume?

3 **Do not use a GDC for this question.**

An open rectangular box is to be made from a piece of card 16cm by 24cm by cutting out square corners and folding it up. What is the maximum possible volume of the box?

4 You need to cross a field 100m by 100m from corner to corner. You can jog at 3ms^{-1} round the outside of the field or through the field at 2ms^{-1}. What is the best route to take?

5 **Do not use a calculator for this question.**

A company produces electrical components. The cost of pricing x components is given by $C(x) = 0.2x^3 - 240x + 3000$. What number of components will minimise the cost?

6 **Do not use a calculator for this question.**

A train journey costs $200. The capacity of the train is 150 passengers but they find they are only averaging 100 passengers. Each reduction of $5 in the price will increase the average number of passengers by 3. What is the best ticket price?

7 **Do not use a calculator for this question.**

The displacement of a particle is given by $s = 98 + 84t - 14t^2, t > 0$. Find

a The velocity of the particle

b The maximum displacement

c The velocity of the particle when the displacement is 0.

8 A particle moves so that its displacement is given by $s(t) = t^3 - 10t^2 + 12t - 8; 0 \le t \le 10$. Find

a The velocity and acceleration.

b When the particle is at rest.

c When the acceleration is positive.

d The total distance travelled.

Answers

1 Area $= x(160 - 2x)$ where x = width

$\frac{dA}{dx} = 160 - 4x$

$= 0$ when x=40.

this is clearly a maximum

So maximum area = $40 \times 80 = 3200\text{m}^2$.

2 Area $= 2\pi r^2 + 2\pi rh = 10\pi$

$\Rightarrow h = \frac{5 - r^2}{r}$

So Volume $= \pi r^2 \times \frac{5 - r^2}{r} = \pi r\left(5 - r^2\right)$

$\frac{dV}{dr} = 5\pi - 15\pi r^2 = 0$ when $r = \frac{1}{\sqrt{3}}$

Clearly a maximum because $\frac{d^2V}{dr^2} = -30\pi r < 0$

Volume $= \pi \times \frac{1}{3} \times \frac{\frac{13}{3}}{\frac{1}{\sqrt{3}}} = \frac{13\sqrt{3}\pi}{9}$

3 Volume $= x(16 - 2x)(24 - 2x)$

$= 4x^3 - 80x^2 + 384x$

$\frac{dV}{dx} = 12x^2 - 160x + 384$

$= 0$ when $x = 3.139....,10.19...,$

Clearly $x < 8$ and $x = 3.139...$is a maximum

because $\frac{d^2V}{dx^2} = 24x - 160 < 0$

So volume = $540.8... \approx 541\text{cm}^2$.

4 If you jog along the edge for x metres and then cut across diagonally then time taken is

$$t = \frac{x}{3} + \frac{\sqrt{10000 + (100 - x)^2}}{2} = \frac{2x + 3\sqrt{20000 - 200x + x^2}}{6}$$

given by $\frac{dt}{dx} = \frac{2 + \frac{3}{2}\left(20000 - 200x + x^2\right)^{-\frac{1}{2}} \times \left(2x - 200\right)}{6}$

$x = 10.557...$ from GDC

So time $= 70.60... \approx 70.6\text{s}$

5 $C'(x) = 0.6x^2 - 240$

So $C'(x) = 0$ when $x = 20$ and this is a minimum because $C'' < 0$

6 Income from tickets = $(100+x)(200-\frac{5}{3}x)$

$= 20000 + \frac{100}{3}x - \frac{5}{3}x^2$

$\frac{dI}{dx} = \frac{100}{3} - \frac{10}{3}x = 0$ when $x = 10$

and this is a maximum because $\frac{d^2I}{dx^2} < 0$

7 **a** $v = 84 - 28t$

b Maximum displacement occurs when $v = 0$

So $t = 3, s = 224$

c Displacement =0 when $98 + 84t - 14t^2 = 0$

$\Rightarrow t^2 - 6t - 7 = 0$
$\Rightarrow t = 7, v = -112$

8 **a** $v = 3t^2 - 20t + 12$

$a = 6t - 20$

b $v = 3t^2 - 20t + 12 = (3t-2)(t-6)$

$= 0$ when $t = \frac{2}{3}, 6$

c $a > 0$ when t>$\frac{10}{3}$.

d When $t = 0, s = -8$

When $t = \frac{2}{3}, s = -\frac{112}{27}$
When $t = 6, s = -80$
When $t = 10, s = 112$

so total distance travelled=$\left(-\frac{112}{27}+8\right)+\left(-\frac{112}{27}+80\right)+192 = \frac{736}{27}$

4.6 Implicit differentiation and related rates

1 Find $\frac{dy}{dx}$ by implicit differentiation in each case.

a $3x^2 - 5y^2 = 7$

b $x^2 - 2xy + 2y^2 = 11$

c $x = \sqrt{3x^2 - 4y^4}$

2 Find the tangent and normal to the curve $\frac{1}{x} + \frac{1}{y+3} = 1$ at the point (2,-1).

3 Given the curve $x - y = x^2 - xy - y^2$.

a Find $\frac{dy}{dx}$.

b Show that $(1 - x - 2y)\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - 2$

4 Write an equation involving the rates of change of the variables in the following equations.

a Volume of cube = x^3.

b Volume of pyramid = $\frac{1}{3}\pi r^2 h$.

5 The volume of a sphere is increasing at a rate of $3\text{m}^3\text{s}^{-1}$. Find the rate at which the surface area is increasing when the sphere has a volume of 100m^3.

6 An aeroplane is flying horizontally at a speed of 100ms^{-1} and is at a height of 500m. It passes vertically above a point A on the ground. When it has flown a further 2km, find the rate of change of its distance from A.

Answers

1 **a** $6x - 10y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x}{5y}$

b $2x - 2x\dfrac{dy}{dx} - 2y + 4y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x-2y}{2x-4y} = \dfrac{x-y}{x-2y}$

c $1 = \dfrac{1}{2}\left(3x^2 - 4y^4\right)^{-\frac{1}{2}} \times 6x - 16y^3\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\sqrt{3x^2 - 4y^4}}{3x - 8y^3}$

2 $-x^{-2} - \left(y+3\right)^{-2}\dfrac{dy}{dx} = 0$

$\Rightarrow -\dfrac{1}{4} - \dfrac{1}{4}\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = 1$

Equation of tangent is $y = x - 3$

Equation of normal is $y = -x - 1$

3 **a** $1 - \dfrac{dy}{dx} = 2x - x\dfrac{dy}{dx} - y - 2y\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1-2x+y}{1-x-2y}$

b $\left(1-x-2y\right)\dfrac{dy}{dx} = 1 - 2x + y$

$\Rightarrow \left(1-x-2y\right)\dfrac{d^2y}{dx^2} + \left(-1-2\dfrac{dy}{dx}\right)\dfrac{dy}{dx} = -2 + \dfrac{dy}{dx}$

$\Rightarrow \left(1-x-2y\right)\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2 + 2\dfrac{dy}{dx} - 2$

4 **a** $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$

b $\dfrac{dV}{dt} = \dfrac{2}{3}\pi rh\dfrac{dr}{dt} + \dfrac{1}{3}\pi r^2\dfrac{dh}{dt}$

5 $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$

When V=100, r =2.879...

So $\dfrac{dr}{dt} = 0.02879...$

$\dfrac{dA}{dt} = 8\pi r\dfrac{dr}{dt} = 2.083... \approx 2.08\text{cm}^2\text{s}^{-1}$

6

$D = \sqrt{500^2 + x^2}$

$\dfrac{dD}{dt} = \dfrac{1}{2}\left(500^2 + x^2\right)^{-\frac{1}{2}} \times 2x\dfrac{dx}{dt} = \dfrac{x}{\sqrt{500^2 + x^2}}\dfrac{dx}{dt}$

$= \dfrac{2000}{\sqrt{500^2 + 2000^2}} \times 100 = 97.01... \approx 97.0\text{ms}^{-1}$

5.1 Sampling

1 For each of **a** and **b** define:

i the target population

ii the sampling unit

iii the sampling frame

iv the sampling variable

v the sampling value

a An investigation into the heights in cm of adult males in France

b The weight, to the nearest gram, of 1kg bags of sugar that a sugar packing plant produces.

2 A country claims that the life expectancy of its adult females is 81 years. Explain why testing the population is not possible. Suggest a sampling technique which could be beneficial to test this claim.

3 Explain why conducting a poll by email might introduce bias.

Answers

1 **a** **i** All the adult males in France

ii Each adult male in France

iii The list of all adult males in France

iv The height of the adult male in cm.

v A reasonable range – say 100-250cm

b **i** All the bags of sugar produced by the plant

ii Each bag of sugar produced by the plant

iii The list of all the bags of sugar produced by the plant

iv The weight of each bag of sugar to the nearest gram

v A reasonable range, say 900-1100g

2 It would be logistically impossible to do this for various reasons including the fact that getting data on all the deaths in a given time frame would be very difficult. Taking a good sized stratified sample covering various target groups could help to test the claim.

3 Various problems including the fact it excludes people who do not use email or have access to the internet.

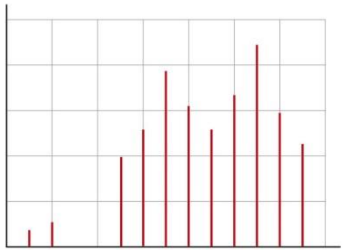
5.2 Descriptive statistics

1 The weights of 40 carrots picked from a farm were measured and recorded to the nearest gram. The results are shown in this frequency table:

Weight to the nearest g	$30 \le x < 40$	$40 \le x < 50$	$50 \le x < 60$	$60 \le x < 70$	$70 \le x < 80$
Frequency	4	7	18	8	3

Draw a frequency histogram to illustrate the data

2 Describe this distribution:



3 Draw a histogram for the data in the table, which represents the weights in kg of 392 parcels sent via a courier postal service in a month.

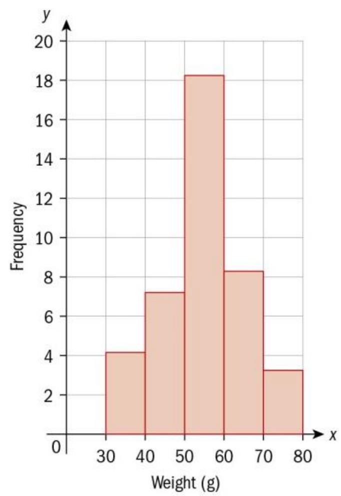
Weight (in kg)	Frequency
$0 \le x < 5$	2
$10 \le x < 15$	17
$15 \le x < 20$	35
$20 \le x < 25$	70
$25 \le x < 30$	50
$30 \le x < 35$	42
$35 \le x < 40$	37
$40 \le x < 45$	65
$45 \le x < 50$	35
$50 \le x < 55$	21
$55 \le x < 60$	18

4 Calculate an estimate of the mean for the data in the table, which represent the heights of 50 children in a primary school class.

Height (in cm)	Frequency
$100 \le x < 110$	3
$110 \le x < 120$	8
$120 \le x < 125$	10
$125 \le x < 130$	11
$130 \le x < 140$	12
$140 \le x < 150$	6

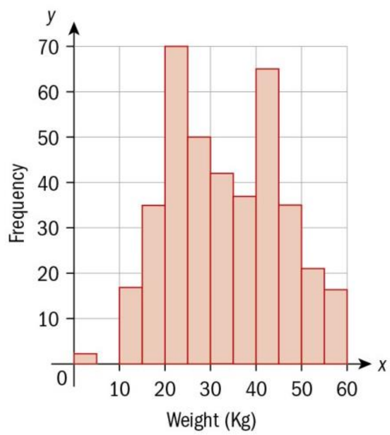
Answers

1



2 Bimodal, fairly symmetric but with outliers

3



4 127.05cm

5.3 The justification of statistical techniques

1 Find the mean and standard deviation of the following:

a 14m, 14m, 18m, 19m, 26m, 29m, 32m

b

x	2 - 4	5 - 7	8 - 10	11 - 13
Frequency	6	5	9	1

2 The average height of tomato plants in a greenhouse in 2017 was 43.1 cm. In 2018 a new fertiliser was used and the results were as follows:

Height in cm	Frequency
20-30	3
30-40	5
40-50	10
50-60	11
60-70	14

a Calculate an estimate of the mean and standard deviation. Explain why these are only estimates.

b Comment on whether or not you think there has been an increase in the height of the tomato plants.

3 Jim takes 10 samples to measure the nitrate concentration in the soil in his garden. The results were as follows where x is in ppm (parts per million)

$\sum x = 63$, $\sum x^2 = 478$

Calculate an estimate of the mean and standard deviation of the nitrate concentration in the garden.

Answers

1 **a** Mean $= \dfrac{14+14+18+19+26+29+32}{7} = \dfrac{152}{7} = 21.71$

Standard deviation $= \sqrt{\dfrac{14^2+14^2+18^2+19^2+26^2+29^2+32^2}{7} - \left(\dfrac{152}{7}\right)^2} = \sqrt{\dfrac{2222}{49}} = 6.73$

b Mean $= \dfrac{3\times 6+6\times 5+9\times 9+12\times 1}{6+5+9+1} = \dfrac{141}{21} = 6.71$

Standard deviation $= \sqrt{\dfrac{3^2\times 6+6^2\times 5+9^2\times 9+12^2\times 1}{6+5+9+1} - \left(\dfrac{141}{21}\right)^2} = \sqrt{\dfrac{374}{49}} = 2.76$

2 **a** Mean $= \dfrac{25\times 3+35\times 5+45\times 10+55\times 11+65\times 14}{3+5+10+11+14} = \dfrac{2215}{43} = 51.51$

Standard deviation

$$= \sqrt{\dfrac{25^2\times 3+35^2\times 5+45^2\times 10+55^2\times 11+65^2\times 14}{3+5+10+11+14} - \left(\dfrac{2215}{43}\right)^2} = \sqrt{\dfrac{282800}{1849}} = 12.37$$

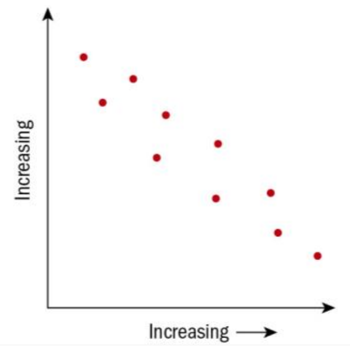
b Although the mean has increased, it is not clear whether this increase is significant- it is less than the standard deviation.

3 Mean $= \dfrac{63}{10} = 6.3$

Standard deviation $= \sqrt{\dfrac{478}{10} - 6.3^2} = \dfrac{\sqrt{811}}{10} = 2.85$

5.4 Correlation, causation and linear regreession

1 This scatter graph shows maths grade (vertical) against the number of hours spent watching television each week. Comment on the graph.



2 This table gives the mathematics and physics results in 2 tests given to 25 students:

Mathematics score	Physics score
20	18
22	28
25	23
26	31
27	24
27	29
28	30
31	31
32	22
33	40
34	32
34	34
35	38
36	41
37	34
38	36
40	47
42	41
44	43
45	48
45	43
46	32
47	43
49	41
50	48

a Calculate the line of best fit.

b Comment on the correlation coefficient.

c Predict the physics score of a student who achieves 34 in mathematics.

Answers

1 There is a weak negative correlation.

2 **a** $y = 0.947x + 2.12$

b 0.744 so strong positive linear correlation

c $y = 0.947 \times 34 + 2.12 = 34.3 \approx 34$

6.1 The properties of three-dimensional space

1 Find the mid-point of each pair of points.

a $(-1,2,7),(6,4,3)$

b $(2,-3,4),(-1,-7,1)$

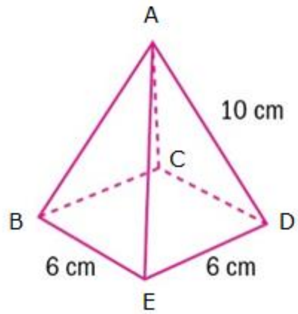
2 Find the distance between each pair of points in question 1.

3 Consider a cuboid with edges of length 5cm, 6cm and 7cm.

a Find the length of the diagonal of the cuboid.

b Find the angle that the diagonal makes with the 5cm by 6cm face.

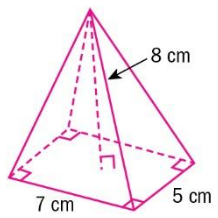
4 ABCDE is a square-based pyramid as shown in the diagram below.



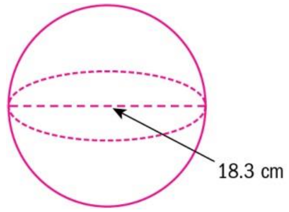
Find the angle that the edge AB makes with the base.

5 Find the volume and surface area of these two shapes.

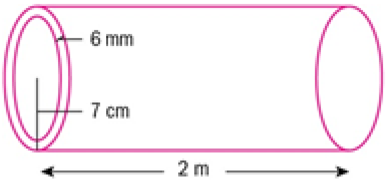
a



b



6 A pipe of thickness 6mm and outer radius 7cm is 2m long.



Find the volume of material required to make the pipe.

Answers

1 **a** $\left(2\frac{1}{2}, 3, 5\right)$

b $\left(\frac{1}{2}, -5, 2\frac{1}{2}\right)$

2 **a** $\sqrt{7^2+2^2+4^2} = 8.306.... \approx 8.31$

b $\sqrt{3^2+4^2+3^2} = 5.830.... \approx 5.83$

3 **a** $\sqrt{5^2+6^2+7^2} = 10.48.... \approx 10.5\text{cm}$

b $\sqrt{5^2+6^2} = 7.810...$

So $\cos\theta = \dfrac{7.810...}{10.48...}$

$\Rightarrow \theta = 41.86... \approx 41.9^\circ$

4 Angle = $\arccos\left(\dfrac{\frac{1}{2}\sqrt{6^2+6^2}}{10}\right)$

$= 64.89..... \approx 64.9^\circ$

5 **a** Volume = $\frac{1}{3}\times 5\times 7\times 8$

$= 93.33... \approx 93.3\text{cm}^3$

Surface area = $(5\times 7)+2\times\left(\frac{1}{2}\times 5\times\sqrt{3.5^2+8^2}\right)+2\times\left(\frac{1}{2}\times 7\times\sqrt{2.5^2+8^2}\right)$

$= 137.3 \approx 137\text{cm}^2$

b Volume = $\frac{4}{3}\pi\times 9.15^3$

$= 3208.8... \approx 3210\text{cm}^3$

Surface area = $4\pi\times 9.15^2$

$= 1052.0... \approx 1050\text{cm}^2$

6 Volume = $\left(\pi\times 7^2\times 200\right)-\left(\pi\times 6.4^2\times 200\right)$

$= 5051.6... \approx 5050\text{cm}^3$

6.2 Angles of measure

1 **Do not use a calculator for this question.**
Express in radians

a 20°

b 90°

c 240°

d 675°

2 **Do not use a calculator for this question.**
Express in degrees

a $\frac{\pi}{4}$

b $\frac{\pi}{20}$

c $\frac{17\pi}{9}$

d $\frac{55\pi}{18}$

3 Express in radians correct to 3 significant figures

a 11°

b 173°

c 229°

4 Express in degrees correct to 3 significant figures

a 1.5^c

b 2.71^c

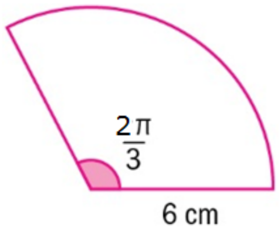
c 4.62^c

5 **Do not use a calculator for this question.**
For each sector, find

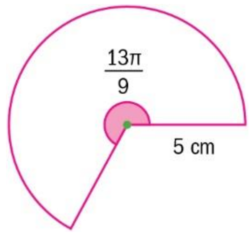
i the arc length

ii the area

a



b



6 **Do not use a calculator for this question.**

A sector has a radius of 10cm and a perimeter of $20+9\pi\text{cm}$. Find

a the angle of the sector

b the area of the sector.

Answers

1 **a** $\frac{\pi}{9}$

b $\frac{\pi}{4}$

c $\frac{4\pi}{3}$

d $\frac{15\pi}{4}$

2 **a** 45°

b 9°

c 340°

d 550°

3 **a** $0.1919\ldots \approx 0.192^c$.

b $3.019\ldots \approx 3.02^c$.

c $3.996\ldots \approx 4.00^c$.

4 **a** $85.94\ldots \approx 85.9^\circ$

b $154.6\ldots \approx 155^\circ$

c $263.7\ldots \approx 264^\circ$

5 **a** **i** $12+6\times\frac{2\pi}{3}=12+4\pi\text{cm}$

a **ii** $\frac{1}{2}\times 6^2\times\frac{2\pi}{3}=12\pi\text{cm}^2$

b **i** $10+5\times\frac{13\pi}{9}=10+\frac{65\pi}{9}\text{cm}$

b **ii** $\frac{1}{2}\times 5^2\times\frac{13\pi}{9}=\frac{325\pi}{18}\text{cm}^2$

6 **a** $\theta=\frac{9\pi}{10}$.

b $\text{Area} = \frac{1}{2}\times 10^2\times\frac{9\pi}{10}=45\pi\text{cm}^2$.

6.3 Ratios and identities

1 Do not use a calculator for this question.

Write down the quadrant and associated angle when θ is

a -240°

b $\frac{7\pi}{4}$

c $-\frac{8\pi}{3}$

d 687°

2 Find all values of θ between 0° and 360° such that

a $\cos\theta = 0.3$

b $\tan\theta = -1.7$

3 Do not use a calculator for this question.

Given that θ is obtuse and $\tan\theta = -\frac{5}{12}$, find $\sin\theta$ and $\cos\theta$.

4 For each of the following, find the values of θ for $0 < \theta < 3$.

a $\cos\theta = -0.73$

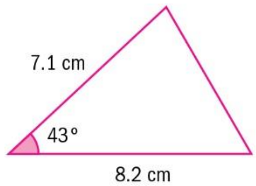
b $\sin\theta = -2.1$

c $3\cos\theta = 2$

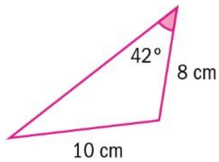
d $\tan 1.5\theta = 2$

5 Use the cosine rule to find the angles in a triangle with sides 3cm, 5cm and 6 cm.

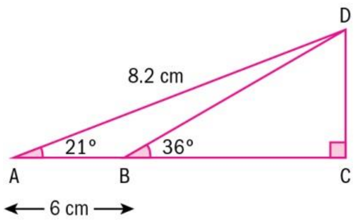
6 Find the third side in this triangle.



7 Find the unknown sides and angles in this triangle.



8 Find the length CD in this diagram



9 In triangle ABC, $\angle ABC = 21^\circ$, $AC = 12\text{cm}$ and $BC = 20\text{cm}$. Find the areas of the two possible triangles.

10 Two planes leave an airport at the same time. One flies at 400kmh^{-1} on a bearing of 043° and the other flies at 300kmh^{-1}on a bearing of 327. How far apart are they after one hour and what is the bearing of the first plane from the second plane?

11 Do not use a calculator for this question.

Evaluate exactly $\sec\frac{3\pi}{4} - \tan\frac{7\pi}{6}$.

12 Simplify to give a single trigonometric expression.

a $\operatorname{cosec}^2\theta\cos\theta\sin\theta$

b $\dfrac{\tan\theta}{\sin\theta}$

13 Given that cosecθ =1.3 and $\frac{\pi}{2} < \theta < \pi$, find the values of cosθ, sinθ and tanθ.

14 Solve the following equations for $0 \le \theta \le 2\pi$.

a $2 - 2\cos\theta = \sin^2\theta$

b $\operatorname{cosec}^2\theta = 1 + \cot\theta$

c $2\tan^2\theta + 5 = 5\sec\theta$

15 Simplify

a $(\sec\theta - 1)(\sec\theta + 1)$

b $\dfrac{2\cot\theta + \operatorname{cosec}^2\theta}{1 + \cot\theta}$

16 Do not use a calculator for this question.

Evaluate exactly: **a** $\sin 15^\circ$ **b** $\tan 165^\circ$.

17 Prove the following identities:

a $\operatorname{cosec}A + \cot A = \dfrac{1 + \cos A}{\sin A}$

b $\tan x - \cot x = -2\tan 2x$

c $\sec^4 x - \tan^4 x = \sec^2 x(1 + \sin^2 x)$

18 Show that $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

19 Use the fact that $\cos 3x = 4\cos^3 x - 3\cos x$ to solve the equation $\cos 3x = \cos^2 x$ for $0 \le x \le 2\pi$.

Answers

1 **a** Q II 60°

b Q IV $\frac{\pi}{4}$

c Q III $\frac{\pi}{3}$

d Q IV 33°

2 **a** $\arccos(0.3) = 72.54...$

$\Rightarrow \theta = 72.54..., 360 - 72.54...$

$\approx 72.5^\circ, 287.5^\circ$

b $\arctan(1.7) = 59.53...$

$\Rightarrow \theta = 180 - 59.53..., 360 - 59.53...$

$\approx 120.5^\circ, 300.5^\circ$

3 In QI by Pythagoras,

$$\tan\theta = \frac{5}{12} \Rightarrow \cos\theta = \frac{12}{13}, \sin\theta = \frac{5}{13}$$

So in QII

$$\cos\theta = -\frac{12}{13}, \sin\theta = \frac{5}{13}$$

4 **a** $\arccos(0.73) = 0.7524...$

$\Rightarrow \theta = \pi - 0.7524... = 2.389...$

$\approx 2.39^c$

b $\arcsin(2.1)$ is undefined

$\Rightarrow$ no values of θ.

c $3\cos\theta = 2 \Rightarrow \cos\theta = \pm\frac{2}{3}$

$$\arccos\left(\frac{2}{3}\right) = 0.8410...$$

$\Rightarrow \theta = 0.8410..., \pi - 0.8410....$

$\approx 0.841^c, 2.30^c$

d $\arctan(2) = 1.107...$

$\Rightarrow 1.5\theta = 1.107..., \pi + 1.107...$

$\Rightarrow \theta = 0.7380...., 2.832....$

$\approx 0.738^c, 2.83^c$

5 One angle = $\arccos\left(\dfrac{3^2+5^2-6^2}{2\times3\times5}\right) = 93.82...$

$\approx 93.8^\circ$

Another angle = $\arccos\left(\dfrac{3^2+6^2-5^2}{2\times3\times6}\right) = 56.25....$

$\approx 56.3^\circ$

So third angle =29.93...

$\approx 29.9^\circ$

6 $\sqrt{7.1^2+8.2^2-2\times7.1\times8.2\times\cos 43^\circ}$

$= 5.700...$

$\approx 5.70cm$

7 Angle opposite 8cm side: $\dfrac{\sin x}{8} = \dfrac{\sin 42}{10}$

$\Rightarrow x = \arcsin\left(\dfrac{8\sin 42}{10}\right) = 32.36...$

$\approx 32.4^\circ$

So third angle $\approx 105.6^\circ$

Third side=$\sqrt{8^2+10^2-2\times8\times10\cos 105.6...}$

$= 14.39... \approx 14.4$cm

8 Find BC first

$CD = (BC+6)\tan 21$

$CD = BC\tan 36$

$\Rightarrow BC = \dfrac{6\tan 21}{\tan 36 - \tan 21}$

So $CD = \dfrac{6\tan 21\tan 36}{\tan 36 - \tan 21} = 4.883... \approx 4.88$cm

9 $\frac{\sin 21}{12} = \frac{\sin A}{20}$

$\Rightarrow \sin A = \frac{20\sin 21}{12}$

$\Rightarrow A = 36.67\ldots \text{or} 143.32\ldots$

$\Rightarrow C = 122.3\ldots \text{or } 15.68\ldots$

Area = $\frac{1}{2} \times 12 \times 20 \sin C = 101.4\ldots, 32.43\ldots$

$\approx 101\text{cm}^2, 32.4\text{cm}^2$

10 Position of first plane = $(400\cos 43, 400\sin 43)$

Position of second plane = $(-300\cos 33, 300\sin 33)$

So distance=$\sqrt{(400\cos 43 + 300\cos 33)^2 + (400\sin 43 - 300\sin 33)^2}$

$= 555.0\ldots \approx 555\text{km}$

Bearing = $\arctan\left(\frac{400\cos 43 + 300\cos 33}{400\sin 43 - 300\sin 33}\right)$

$= 78.63\ldots \approx 078.6^\circ$

11 $\sec\frac{3\pi}{4} - \tan\frac{7\pi}{6} = \frac{1}{\cos\frac{3\pi}{4}} - \left(\tan\frac{\pi}{6}\right)$

$= \frac{1}{-\cos\frac{\pi}{4}} - \left(\frac{1}{\sqrt{3}}\right)$

$= -\sqrt{2} - \frac{1}{\sqrt{3}}.$

12 **a** $\operatorname{cosec}^2\theta \cos\theta \sin\theta$

$= \frac{\cos\theta \sin\theta}{\sin^2\theta}$

$= \frac{\cos\theta}{\sin\theta}$

$= \cot\theta$

b $$\frac{\tan\theta}{\sin\theta}=\frac{\sin\theta}{\cos\theta\sin\theta}$$

$$=\frac{1}{\cos\theta}$$

$$=\sec\theta$$

13 $$\operatorname{cosec}\theta=1.3\Rightarrow\sin\theta=\frac{1}{1.3}=0.7692...\approx0.769$$

$$\Rightarrow\cos\theta=-\sqrt{1-\left(\frac{1}{1.3}\right)^2}\text{ since }\frac{\pi}{2}<\theta<\pi$$

$$=-0.6389...\approx-0.639.$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{0.7692...}{-0.6389...}=-1.203...\approx1.20$$

14 **a** $2-2\cos\theta=\sin^2\theta$

$$\Rightarrow2-2\cos\theta=1-\cos^2\theta$$

$$\Rightarrow\cos^2\theta-2\cos\theta+1=0$$

$$\Rightarrow(\cos\theta-1)^2=0$$

$$\Rightarrow\cos\theta=1$$

$$\Rightarrow\theta=0,2\pi$$

b $\operatorname{cosec}^2\theta=1+\cot\theta$

$$\Rightarrow\cot^2\theta+1=\cot\theta+1$$

$$\Rightarrow\cot\theta(\cot\theta-1)=0$$

$$\Rightarrow\cot\theta=0,1$$

$$\Rightarrow\theta=\frac{\pi}{4},\frac{\pi}{2},\frac{5\pi}{4},\frac{3\pi}{2}$$

c $2\tan^2\theta+5=5\sec\theta$

$$\Rightarrow2(\sec^2\theta-1)+5=5\sec\theta$$

$$\Rightarrow2\sec^2\theta-5\sec\theta+3=0$$

$$\Rightarrow(2\sec\theta-3)(\sec\theta-1)=0$$

$$\Rightarrow\sec\theta=1,1.5$$

$$\Rightarrow\cos\theta=1,\frac{2}{3}$$

$$\Rightarrow\theta=0,0.8410...,2\pi-0.8410...,2\pi$$

$$0,0.841,5.44,2\pi$$

15 **a** $(\sec\theta-1)(\sec\theta+1)$

$=\sec^2\theta-1$

$=\tan^2\theta$

b $\dfrac{2\cot\theta+\operatorname{cosec}^2\theta}{1+\cot\theta}$

$=\dfrac{2\cot\theta+\cot^2\theta+1}{1+\cot\theta}$

$=\dfrac{(1+\cot\theta)^2}{1+\cot\theta}$

$=1+\cot\theta.$

16 **a** $\sin 15^\circ=\sin\left(45^\circ-30^\circ\right)$

$=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ$

$=\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2}-\dfrac{1}{\sqrt{2}}\times\dfrac{1}{2}$

$=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$

b $\tan 165^\circ=\tan\left(120^\circ+45^\circ\right)$

$=\dfrac{\tan 120^\circ+\tan 45^\circ}{1-\tan 120^\circ\tan 45^\circ}$

$=\dfrac{-\sqrt{3}+1}{1-\left(-\sqrt{3}\right)}$

$=\dfrac{1-\sqrt{3}}{1+\sqrt{3}}$

17 **a** $\operatorname{cosec}A+\cot A=\dfrac{1}{\sin A}+\dfrac{\cos A}{\sin A}=\dfrac{1+\cos A}{\sin A}$

b $\tan x-\cot x$

$=\dfrac{\sin x}{\cos x}-\dfrac{\cos x}{\sin x}$

$=\dfrac{\sin^2 x-\cos^2 x}{\cos x\sin x}$

$=\dfrac{-\cos 2x}{\frac{1}{2}\sin 2x}$

$=-2\tan 2x$

c $\sec^4 x-\tan^4 x$

$$=\left(\sec^2 x-\tan^2 x\right)\left(\sec^2 x+\tan^2 x\right)$$
$$=1\times\left(\sec^2 x+\tan^2 x\right)$$
$$=\frac{1}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}$$
$$=\frac{1}{\cos^2 x}\left(1+\sin^2 x\right)$$
$$=\sec^2 x\left(1+\sin^2 x\right)$$

18 $\tan 3x$

$$=\tan\left(2x+x\right)$$
$$=\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}$$
$$=\frac{\frac{2\tan x}{1-\tan^2 x}+\tan x}{1-\frac{2\tan x}{1-\tan^2 x}\tan x}$$
$$=\frac{2\tan x+\tan x\left(1-\tan^2 x\right)}{\left(1-\tan^2 x\right)-2\tan^2 x}$$
$$=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$$

19 $4\cos^3 x-3\cos x=\cos^2 x$

$$\Rightarrow \cos x\left(4\cos^2 x-\cos x-3\right)=0$$
$$\Rightarrow \cos x\left(4\cos x+3\right)\left(\cos x-1\right)$$
$$\cos x=0,-\frac{3}{4},1$$
$$x=0,\frac{\pi}{2},2.42,3.86,\frac{3\pi}{2},2\pi$$

6.4 Trigonometric functions

1 Use the graph $f(\theta)=\cos\theta$ to deduce the graph and properties of $f(\theta)=\sec\theta$.

2 Use transformations of $y=\sin x$ to graph each of the following functions

a $y=2\sin 3x$

b $y=3\sin(2x-\pi)+2$

3 The height h in metres of the tide in a harbour port on a certain day is given by the equation $h=7.2\cos(0.26(t+3.1))+10$ where t is the time in hours after midnight.

a Find the time of the first high tide.

b When is the tide lower than 6m?

4 Write down the values of the following angles

a $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

b $\arctan(-1)$

c $\arccos\left(-\frac{1}{\sqrt{2}}\right)$

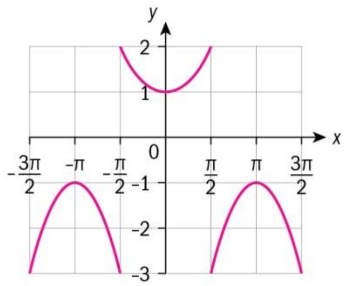
5 For the graph $y=2\cos 3x$, find

a The period

b The amplitude.

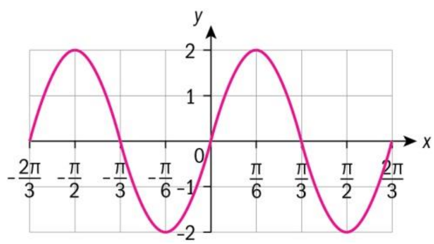
Answers

1

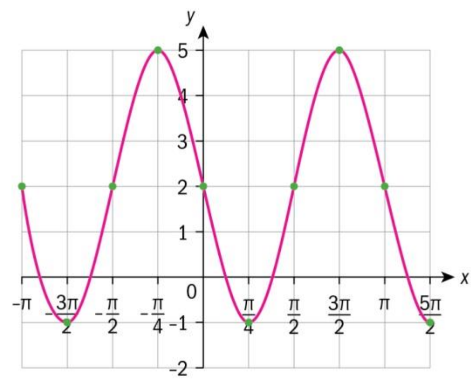


2

a



b



3 **a**

High tide is 17.2

$7.2\cos\left(0.26\left(t+3.1\right)\right)+10=17.2$

$\Rightarrow \cos\left(0.26\left(t+3.1\right)\right)=1$

$\Rightarrow 0.26\left(t+3.1\right)=0, 2\pi, \ldots$

$\Rightarrow t=21.06\ldots$

$\Rightarrow$ time is 21 : 04

b

$7.2\cos\left(0.26\left(t+3.1\right)\right)+10<6$

$\Rightarrow \cos\left(0.26\left(t+3.1\right)\right)<-0.5555\ldots$

$\cos\left(0.26\left(t+3.1\right)\right)<-0.5555\ldots$ when $\left(0.26\left(t+3.1\right)\right)<2.159\ldots, 4.123\ldots, 8.443\ldots$

$\Rightarrow$ Between 05:12 and 12:46

4 **a** $\frac{\pi}{3}$ **b** $-\frac{\pi}{4}$ **c** $\frac{3\pi}{4}$

5 **a** $\frac{2\pi}{3}$ **b** 2

6.5 Trigonometric equations

1 Find the roots of the equations for $0 \le \theta \le 2\pi$ and check on a GDC.

a $\sec^2\theta = 4\sec\theta - 4$

b $\sqrt{2}\operatorname{cosec}\theta - \cot\theta = \tan\theta$

2 Prove by differentiation from first principles that

$$\frac{\mathrm{d}(\cos 2x)}{\mathrm{d}x} = -2\sin 2x$$

3 Differentiate with respect to x:

a $\sin 3x$

b $\tan\left(\frac{2-3x}{7}\right)$

c $\sin\left(2x^2-1\right)$

d $\operatorname{cosec}\sqrt{x^2-3}$

4 Differentiate with respect to x:

a $(2x-3)\cos x$

b $\frac{3x-7}{\sin 4x}$

5 Find the gradient of the curve at the given point.

a $y = \cos 2x$ at $x = \frac{\pi}{12}$

b $y = x^4 \sec x$ at $x = \frac{\pi}{4}$

6 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \arcsin 3x$.

7 Find the equation of the tangent to the curve $y = \left(1-x^2\right)\arccos x$ when $x = 0$.

8 A triangle has sides 8cm, 8cm and xcm. The angle between the two 8 cm sides is denoted by A.

a Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}t}$ in terms of $\frac{\mathrm{d}x}{\mathrm{d}t}$

b If $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.1\,\mathrm{cms}^{-1}$, find an expression for $\frac{\mathrm{d}A}{\mathrm{d}t}$ when the triangle is equilateral.

9 A ladder of length 7m is leaning against a vertical wall when it starts to slip down at a rate of $0.4\mathrm{ms}^{-1}$. Find the rate of change of x, the distance of the foot of the ladder from the wall, when the top of the ladder is 6cm above the ground.

Answers

1 **a** $\sec^2\theta = 4\sec\theta - 4$

$\Rightarrow (\sec\theta - 2)^2 = 0$

$\Rightarrow \sec\theta = 2$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

b $\sqrt{2}\text{cosec}\theta - \cot\theta = \tan\theta$

$\Rightarrow \frac{\sqrt{2}}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \sqrt{2}\cos\theta - \cos^2\theta = \sin^2\theta$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{7\pi}{4}$

2 $\lim\limits_{h\to 0} \frac{\cos(2(x+h)) - \cos 2x}{h}$

$= \lim\limits_{h\to 0} \frac{\cos 2x \cos 2\text{h} - \sin 2x \sin 2\text{h} - \cos 2x}{h}$

$= \lim\limits_{h\to 0} \left(\cos 2x \frac{\cos 2h - 1}{h} - \sin 2x \frac{\sin 2h}{h} \right)$

$= \cos 2x \lim\limits_{h\to 0} \frac{\cos 2h - 1}{h} - \sin 2x \lim\limits_{h\to 0} \frac{\sin 2h}{h}$

$= \cos 2x \times 0 - \sin 2x \times 2$

$= -2\sin 2x$

3 **a** $3\cos 3x$

b $\sec^2\left(\frac{2-3x}{7}\right) \times \left(-\frac{3}{7}\right)$

$= -\frac{3}{7}\sec^2\left(\frac{2-3x}{7}\right)$

c $4x\cos(2x^2 - 1)$

d $-\operatorname{cosec}\sqrt{x^2-3}\cot\sqrt{x^2-3}\times\frac{1}{2}\left(x^2-3\right)^{-\frac{1}{2}}\times 2x$

$=-\dfrac{x\operatorname{cosec}\sqrt{x^2-3}\cot\sqrt{x^2-3}}{\sqrt{x^2-3}}$

4 **a** $2\cos x-(2x-3)\sin x$

b $\dfrac{3\sin 4x-(3x-7)\times 4\cos 4x}{\sin^2 4x}$

$=\dfrac{3\sin 4x-(12x-28)\cos 4x}{\sin^2 4x}$

5 **a** $\dfrac{dy}{dx}=-2\sin 2x=-2\times\sin\dfrac{\pi}{6}$

$=-1$

b $\dfrac{dy}{dx}=4x^3\sec x+x^4\sec x\tan x$

$=4\left(\dfrac{\pi}{4}\right)^3\sec\left(\dfrac{\pi}{4}\right)+\left(\dfrac{\pi}{4}\right)^4\sec\left(\dfrac{\pi}{4}\right)\tan\left(\dfrac{\pi}{4}\right)$

$=4\times\dfrac{\pi^3}{64}\times\sqrt{2}+4\times\dfrac{\pi^4}{256}\times\sqrt{2}\times 1$

$=\dfrac{\sqrt{2}\pi^3}{16}+\dfrac{\sqrt{2}\pi^4}{64}$

6 $-\dfrac{3}{\sqrt{1-9x^2}}$

7 $\dfrac{dy}{dx}=-2x\arccos x-\dfrac{1-x^2}{\sqrt{1-x^2}}$

$=-2x\arccos x-\sqrt{1-x^2}$

$=-1$

$y=-x+c$

$c=\dfrac{\pi}{2}$

$y=\dfrac{\pi}{2}-x$

8 **a** $A = 2\arcsin\dfrac{\frac{1}{2}x}{8} = 2\arcsin\dfrac{x}{16}$

$$\frac{dA}{dt} = \frac{2}{\sqrt{1-\left(\frac{x}{16}\right)^2}} \times \frac{1}{16}\frac{dx}{dt}$$

$$= \frac{1}{8\sqrt{1-\frac{x^2}{256}}}\frac{dx}{dt}$$

b $\dfrac{dA}{dt} = \dfrac{1}{8\times\sqrt{1-\frac{1}{4}}} \times 0.1$

$$= \frac{1}{40\sqrt{3}}\text{ rad/s}$$

9 $x = \sqrt{49-y^2}$

$$\Rightarrow \frac{dx}{dt} = -2y \times \frac{1}{2}\left(49-y^2\right)^{-\frac{1}{2}}\frac{dy}{dt} = -\frac{y}{\sqrt{49-y^2}}\frac{dy}{dt}$$

$$= -\frac{6}{\sqrt{13}} \times -0.4 = 0.6656\ldots \approx 0.666\text{ms}^{-1}$$

7.1 Integration as antidifferentiation and definite integrals

1 Find indefinite integrals of the following expressions:

a $\frac{3}{5}x$ **b** $2x^{\frac{2}{3}}$ **c** $1-2\sin^2\left(\frac{x}{2}\right)$

d $\left(x^2+2x\right)^3$ **e** $x^2+\sin x-\frac{1}{\sin^2 x}$

2 Find $y=f(x)$ given that $\frac{dy}{dx}=4x^3-2x$ and $f(-2)=18$

3 Find $f(x)$ with the given conditions;

a $f'(x)=2-\sin 2x$, $f\left(\frac{\pi}{4}\right)=\frac{\pi}{2}+1$ **b** $f''(x)=1+4\sin x$, $f'(0)=1$, $f(0)=6$

4 A particle moves in a straight line such that, at time t, its acceleration is $a(t)=16t-4$. When $t=0$, its velocity is 3ms^{-1} and its displacement is 4m. Find expressions for the velocity and displacement of the particle.

5 Integrate with respect to x:

a $(1-4x)^5$ **b** $7(2-3x)^{-\frac{1}{4}}$

6 Use definite integration to find the areas of the regions bounded by the graph of the function, the x-axis and the given lines. Verify your result by finding the area of a geometric shape.

a $y=2x+1$, $x=0$, $x=7$ **b** $y=|x|+1$, $x=-2$, $x=2$

7 Find the areas of the regions bounded by the graph of the function, the x-axis and the given lines. Verify your result using a GDC.

a $y=x^2-3x-4$, $x=-1$, $x=4$ **b** $y=7x-12-x^2$, $x=2$, $x=5$

Answers

1 **a** $\frac{3}{10}x^2+c$ **b** $\frac{6}{5}x^{\frac{5}{3}}+c$

c Since $\cos 2x = 1-2\sin^2 x$, $1-2\sin^2\left(\frac{x}{2}\right)=\cos x$, so the answer is $\sin x + c$

d $\left(x^2+2x\right)^3 = x^6+6x^5+12x^4+8x^3$, so the answer is $\frac{1}{7}x^7+x^6+\frac{12}{5}x^5+2x^4+c$

e The integral of $\text{cosec}^2 x$ is $-\cot x + c$, so the answer is $\frac{1}{3}x^3-\cos x+\cot x+c$

2 $y=x^4-x^2+c$, $18=16-4+c \Rightarrow c=6$, so $y=x^4-x^2+6$

3 **a** $f(x)=2x+\frac{1}{2}\cot 2x+c$, $\frac{\pi}{2}+1=\frac{\pi}{2}+c \Rightarrow c=1$, so $f(x)=2x+\frac{1}{2}\cot 2x+1$

b $f'(x)=x-4\cos x+c$, $1=0-4+c \Rightarrow c=5 \Rightarrow f'(x)=x-4\cos x+5$

$f(x)=\frac{1}{2}x^2-4\sin x+5x+d$, $6=0-0+d \Rightarrow d=6 \Rightarrow f(x)=\frac{1}{2}x^2-4\sin x+5x+6$

4 $v=8t^2-4t+c$, $3=0+c \Rightarrow c=3 \Rightarrow v=8t^2-4t+3$,

$s=\frac{8}{3}t^3-2t^2+3t+d$, $4=0-0+0+d \Rightarrow d=4 \Rightarrow s=\frac{8}{3}t^3-2t^2+3t+4$

5 **a** $-\frac{1}{24}(1-4x)^6+c$

b $-\frac{28}{9}(2-3x)^{\frac{3}{4}}+c$

6 **a** $\int_0^7 2x+1\,dx=\left[x^2+x\right]_0^7=56-0=56$, area of trapezium $=\frac{1}{2}\times(1+15)\times 7=56$

b $\int_{-2}^{2}|x|+1\,dx=\int_{-2}^{0}-x+1\,dx+\int_0^2 x+1\,dx=\left[-\frac{1}{2}x^2+x\right]_{-2}^{0}+\left[\frac{1}{2}x^2+2\right]_0^2=-(-2-2)+(2+2)=8$, area of two trapezia $=2\times\left(\frac{1}{2}\times(1+2)\times 2\right)=8$

7 **a** $\int_{-1}^{4}x^2-3x-4\,dx=\left[\frac{1}{3}x^3-\frac{3}{2}x^2-4x\right]_{-1}^{4}=\left(\frac{64}{3}-24-16\right)-\left(-\frac{1}{3}-\frac{3}{2}+4\right)=-\frac{125}{6}$, so area $=\frac{125}{6}$

b $7x-12-x^2=0$ when $x=3,4$, so area

$$=-\int_2^3 7x-12-x^2\,dx+\int_3^4 7x-12-x^2\,dx-\int_4^5 7x-12-x^2\,dx$$

$$=-\left[\frac{7}{2}x^2-12x-\frac{1}{3}x^3\right]_2^3+\left[\frac{7}{2}x^2-12x-\frac{1}{3}x^3\right]_3^4-\left[\frac{7}{2}x^2-12x-\frac{1}{3}x^3\right]_4^5$$

$$=-\left(-\frac{27}{2}-\left(-\frac{38}{3}\right)\right)+\left(-\frac{40}{3}-\left(-\frac{27}{2}\right)\right)-\left(-\frac{85}{6}-\left(-\frac{40}{3}\right)\right)$$

$$=\frac{11}{6}$$

7.2 Exponents and logarithms

1 Find the value of these expressions:

a $64^{\frac{2}{3}}$ **b** $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$

2 Simplify $35 \times 5^{2n-1} - 2 \times 25^{n}$

3 Solve the following equations:

a $2^{4x} - 2^{5x-7} = 0$ **b** $8^{x} - 5 \times 4^{x} = 4 - 2^{x+3}$

4 A car bought for \$16 000 is worth \$9000 three years later. Find the annual rate of depreciation as a percentage correct to one decimal place.

5 Express in logarithmic form:

a $2^{7} = 128$ **b** $x^{2y} = 14$

6 Express in exponent form:

a $\log_{3}\frac{1}{27} = -3$ **b** $\log_{x} 2 = 5$

7 Solve for x:

a $\log_{x} 81 = 3$ **b** $\log_{8} x = \frac{4}{3}$

8 Express as a single logarithm:

a $\log 12 - 4\log 3 + \log 60$ **b** $\frac{1}{3}\log_{x} y + \frac{2}{3}\log_{x} y^{3}z$

9 Express $\frac{1}{2}\log_{3}\frac{9}{4} + \log_{3} 6 - \frac{1}{3}\log_{3} 81$ as a rational number.

10 Express y in terms of x in each of the following:

a $3\log x = 5\log y$ **b** $\log_{10} y = 1 + 3\log_{10} x$

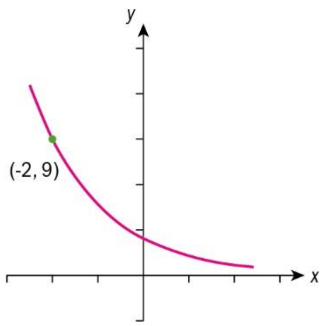
11 Evaluate:

a $\log_{3} 5 \times \log_{5} 3$ **b** $\log_{3} 64 \div \log_{27} 64$ **c** $\log_{4} 80 - \frac{1}{\log_{10} 4}$

12 Solve the equation $2^{4x-1} = 73$

13 A geometric series has first term 5 and common ratio 1.1. find the value of k when the sum of the first k terms of the series first exceeds 10 000.

14 This graph shows an exponential function of the form $y = a^x$. Find the value of a.



15 Solve the equation $e^x + e^{x-1} = 6$

16 Sketch the graphs of:

a $y = \ln 2x$ **b** $y = \ln\left|\frac{x}{3}\right|$

Answers

1 **a** 16 **b** $\frac{8}{27}$

2 $35\times 5^{2n-1}-2\times 25^{n}=7\times 5^{2n}-2\times 5^{2n}=5^{2n+1}$

3 **a** $4x=5x-7\Rightarrow x=7$

b $8^{x}-5\times 4^{x}=4-2^{x+3}\Rightarrow 2^{3x}-5\times 2^{2x}=4-8\times 2^{x}$, let $y=2^{x}$ then we get $y^{3}-5y^{2}+8y-4=0\Rightarrow (y-1)(y^{2}-4y+4)=0\Rightarrow (y-1)(y-2)^{2}=0\Rightarrow y=1,2$ which gives $x=0,1$

4 $\sqrt[3]{\frac{9000}{16000}}=\sqrt[3]{\frac{9}{16}}=0.8253\ldots$, $1-0.8253\ldots=0.175\Rightarrow 17.5\%$

5 **a** $\log_{2}128=7$ **b** $\log_{x}14=2y$

6 **a** $3^{-3}=\frac{1}{27}$ **b** $x^{5}=2$

7 **a** $x^{3}=81\Rightarrow x=\sqrt[3]{81}=3\sqrt[3]{3}$ **b** $8^{\frac{4}{3}}=x\Rightarrow x=16$

8 **a** $\log\frac{12\times 60}{3^{4}}=\log\frac{80}{9}$ **b** $\log_{x}\left(y^{\frac{1}{3}}\times y^{2}z^{\frac{2}{3}}\right)=\log_{x}\left(y^{\frac{7}{3}}z^{\frac{2}{3}}\right)$

9 $\frac{1}{2}\log_{3}\frac{9}{4}+\log_{3}6-\frac{1}{4}\log_{3}81=\log_{3}\frac{\frac{3}{2}\times 6}{3}=\log_{3}3=1$

10 **a** $x^{3}=y^{5}\Rightarrow y=x^{\frac{3}{5}}$

b $\log_{10}y=1+3\log_{10}x\Rightarrow \log_{10}y=\log_{10}10+3\log_{10}x\Rightarrow y=10x^{3}$

11 **a** $\log_{3}5\times\log_{5}3=\log_{3}5\times\frac{\log_{5}5}{\log_{3}5}=\log_{5}5=1$

b $\log_{3}64\div\log_{27}64=\log_{3}64\div\frac{\log_{3}64}{\log_{3}27}=\log_{3}27=3$

c $\log_{4}80-\frac{1}{\log_{10}4}=\log_{4}80-\frac{\log_{4}10}{\log_{4}4}=\log_{4}80-\log_{4}10=\log_{4}8=\frac{3}{2}$

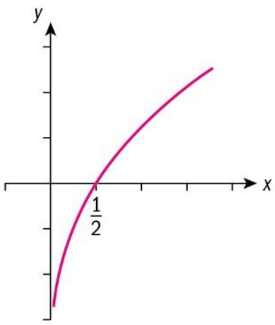
12 $2^{4x-1}=73\Rightarrow (4x-1)\log 2=\log 73\Rightarrow 4x-1=\frac{\log 73}{\log 2}\Rightarrow x=1.797\ldots\approx 1.80$

13 $5\times\frac{1.1^{k}-1}{1.1-1}=10000\Rightarrow 1.1^{k}=\frac{10000}{50}+1\Rightarrow k=\frac{\log 201}{\log 1.1}=55.6\ldots$, so smallest value of k is 55.

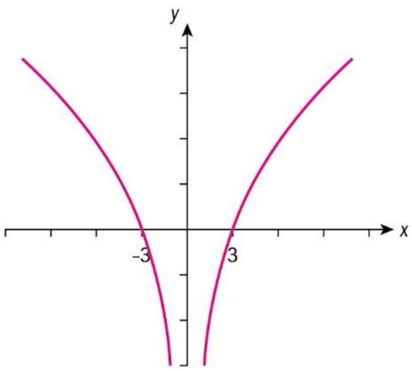
14 $a=\frac{1}{3}$

15 $e^{x}+e^{x-1}=6$, set $y=e^{x}$, then $y+\frac{y}{e}=6\Rightarrow y=\frac{6}{1+\frac{1}{e}}=4.486\ldots\Rightarrow x=\ln 4.486\ldots=1.478\ldots\approx 1.48$

16 **a**



b



7.3 Derivatives of exponential and logarithmic functions; tangents and normals

1 Differentiate the following functions with respect to x:

a $y = 3e^{1-4x}$ **b** $y = e^{2\sin x}$ **c** $y = \ln\left(x^2 - 2x\right)$

2 Use the product rule to find the derivatives of the following functions with respect to x:

a $y = 2xe^{3x-1}$ **b** $y = x^4 \ln(5x+7)$ **c** $y = \dfrac{\cos 2x}{e^{3x}}$

3 Use the quotient rule to find the derivatives of the following functions with respect to x:

a $y = \dfrac{x^2}{e^x}$ **b** $y = \dfrac{1-x^2}{\ln 2x}$ **c** $y = \dfrac{x^2 - e^x}{e^{-2x}}$

4 $f(x) = x^2 - \ln x$

a Show that the function has only one turning point.

b Find the coordinates of the turning point and classify its nature.

c State the domain and range of the function.

d Write the equation of any asymptotes.

e Sketch the curve.

5 $f(x) = e^x \cos 2x$ for $0 \le x \le \pi$

a Show that the function has two turning points and two points of inflexion.

b Identify the nature of the turning points and the coordinates.

c Find the coordinates of the points of inflection.

d Sketch the curve.

Answers

1 **a** $\frac{dy}{dx} = -12e^{1-4x}$ **b** $\frac{dy}{dx} = 2e^{2\sin x}\cos x$ **c** $\frac{dy}{dx} = \frac{2x-2}{x^2-2x}$

2 **a** $\frac{dy}{dx} = 2e^{3x-1} + 6xe^{3x-1} = 2(1+3x)e^{3x-1}$ **b** $\frac{dy}{dx} = 4x^3\ln(5x+7) + \frac{5x^4}{5x+7}$

c $\frac{dy}{dx} = -2e^{-3x}\sin 2x - 3e^{-3x}\cos 2x = -e^{-3x}(2\sin 2x + 3\cos 2x)$

3 **a** $\frac{dy}{dx} = \frac{2xe^x - x^2e^x}{e^{2x}} = \frac{x(2-x)}{e^x}$ **b**

$$\frac{dy}{dx} = \frac{-2x\ln 2x - (1-x^2)\times\frac{2}{2x}}{(\ln 2x)^2} = -\frac{2x^2\ln 2x + 1 - x^2}{x(\ln 2x)^2}$$

c $\frac{dy}{dx} = \frac{(2x-e^x)e^{-2x} - (x^2-e^x)e^{-2x}\times(-2)}{e^{-4x}} = \frac{2x - e^x + 2x^2 - 2e^x}{e^{-2x}} = 2x(x+1)e^{2x} - 3e^{3x}$

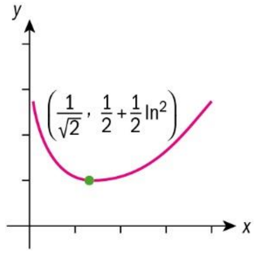
4 **a** $\frac{dy}{dx} = 0 \Rightarrow 2x - \frac{1}{x} = 0 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$, but as $\ln$ is not defined for $x = -\frac{1}{\sqrt{2}}$ so only one turning point

b Coordinates are $\left(\frac{1}{\sqrt{2}}, \frac{1}{2} - \ln\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2} + \frac{1}{2}\ln 2\right)$, $\frac{d^2y}{dx^2} = 2 + \frac{1}{x^2} > 0$ so it is a minimum

c Domain: $x > 0$, range: $y \geq \frac{1}{2} + \frac{1}{2}\ln 2$

d $x = 0$ is an asymptote

e



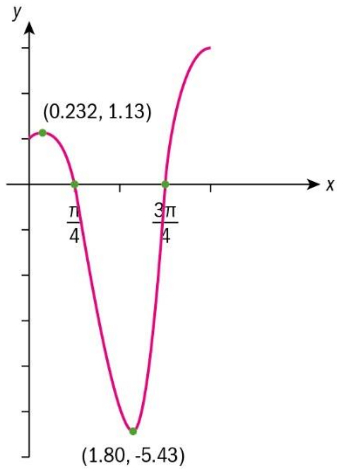
5 **a** $f'(x) = e^x\cos 2x - 2e^x\sin 2x \Rightarrow f'(x) = 0$ when $\cos 2x = 2\sin 2x$ since e^x is never equal to 0, so $\Rightarrow \tan 2x = \frac{1}{2}$ which has 2 roots for $0 \leq x \leq \pi$, so 2 turning points.

$f''(x) = e^x\cos 2x - 2e^x\sin 2x - 2e^x\sin 2x - 4e^x\cos 2x = -3e^x\cos 2x - 4e^x\sin 2x \Rightarrow f''(x) = 0$ when $3\cos 2x = -4\sin 2x$ since e^x is never equal to 0, so $\Rightarrow \tan 2x = -\frac{3}{4}$ which has 2 roots for $0 \leq x \leq \pi$, so 2 points of inflection.

b Using a calculator, we find the coordinates of the turning points are $(0.232, 1.13)$ and $(1.80, -5.43)$. When $x = 0.232$, $f''(x) < 0$ so it is a maximum, when $x = 1.80$, $f''(x) > 0$ so it is a minimum.

c Using a calculator, we find the coordinates of the points of inflection are $(1.25, -2.79)$ and $(2.82, 13.42)$.

d



7.4 Integration techniques

1 Find the following integrals:

a $\int (x^4 + \cos 2x)\, dx$

b $\int (2\sin^2 2x - 1)\, dx$

c $\int \left(\frac{x}{2x-1} + \sin x\right) dx$

d $\int \left(xe^{x^2} - \sec^2 2x\right) dx$

2 Find $f(x)$ given $f'(x) = x^3 - \frac{1}{e^{2x}} + \tan^2 x$ and $f(0) = 2$.

3 Find $f(x)$ given $f''(x) = 2\sin x \cos x$, $f'\left(\frac{\pi}{4}\right) = 3$ and $f(0) = 1$.

4 Evaluate the following definite integrals:

a $\int_0^{\frac{\pi}{6}} (e^{2x-1} - \cos x)\, dx$

b $\int_1^2 \left(2^x - \frac{3}{1-2x}\right) dx$

c $\int_0^2 \frac{3x}{2x+1}\, dx$

d $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos x + \sin x}\, dx$

5 Reduce $\frac{7x+5}{2x^2+3x+1}$, hence find $\int \frac{7x+5}{2x^2+3x+1}\, dx$.

6 Show that $\int_0^1 \frac{2x^2+2x-8}{x^2-4}\, dx = 2 + \ln 3 - 2\ln 2$.

7 Find the following integrals using an appropriate substitution:

a $\int 3x(x^2-7)^2\, dx$

b $\int 2\tan 2x\, dx$

c $\int 2x\sqrt{3x-1}\, dx$

d $\int (x^2 - 2x)\cos(x^3 - 3x^2)\, dx$

e $\int x^3\sqrt{1-x}\, dx$

f $\int x(x+2)^7\, dx$

g $\int \left(\frac{x}{x-3}\right)^2 dx$

h $\int \frac{\cos 2x}{(1+\sin 2x)^3}\, dx$

8 Find the following definite integrals using an appropriate substitution:

a $\int_1^4 \frac{x}{\sqrt{x+1}}\,dx$ **b** $\int_1^3 \frac{4x}{(2x+1)^2}\,dx$ **c** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x}\,dx$ **d** $\int_1^2 2^x\sqrt{2^x}\,dx$

9 Show that $\int \frac{dx}{x^2+2x+2} = \arctan(x+1)+c$ and hence show that $\int_0^{\sqrt{3}-1} \frac{dx}{x^2+2x+2} = \frac{\pi}{12}$

10 Use integration by parts to find the following integrals:

a $\int xe^{3x}\,dx$ **b** $\int (x-1)\cos x\,dx$ **c** $\int x^4 \ln 3x\,dx$ **d** $\int \arctan x\,dx$

e $\int (x^2-x+3)\ln x\,dx$

11 Evaluate the following definite integrals:

a $\int_0^{\frac{1}{\sqrt{2}}} \arcsin x\,dx$ **b** $\int_1^e x^2 \ln x\,dx$ **c** $\int_0^{\frac{\pi}{3}} x\cos x\,dx$

12 Find the following:

a $\int x^2 \cos x\,dx$ **b** $\int (x^2-x+2)e^{3x}\,dx$

13 Evaluate the following definite integrals:

a $\int_0^3 (x^2-1)e^{2x}\,dx$ **b** $\int_0^{\frac{\pi}{3}} x^3\cos x\,dx$ **c** $\int_0^2 \frac{x^2}{e^x}\,dx$

14 Evaluate $\int e^x \sin 3x\,dx$.

Answers

1 **a** $\frac{1}{5}x^5 + \frac{1}{2}\sin 2x + c$

b $2\sin^2 2x - 1 = -\cos 4x \Rightarrow \int\left(2\sin^2 2x - 1\right)dx = -\frac{1}{4}\sin 4x + c$

c $\int\left(\frac{x}{2x-1} + \sin x\right)dx = \int\left(\frac{1}{2} + \frac{\frac{1}{2}}{2x-1} + \sin x\right)dx = \frac{1}{2}x + \frac{1}{4}\ln(2x-1) - \cos x - \frac{1}{4} + c$

d $\frac{1}{2}e^{x^2} - \frac{1}{2}\tan 2x + c$

2 $f'(x) = x^3 - e^{-2x} + \sec^2 x - 1 \Rightarrow f(x) = \frac{1}{4}x^4 + \frac{1}{2}e^{-2x} + \tan x - x + c$, $2 = \frac{1}{2} + c \Rightarrow c = \frac{3}{2}$, so

$f(x) = \frac{1}{4}x^4 + \frac{1}{2}e^{-2x} + \tan x - x + \frac{3}{2}$

3 $f''(x) = \sin 2x \Rightarrow f'(x) = -\frac{1}{2}\cos 2x + c$, $0 + c = 3 \Rightarrow c = 3$, so $f'(x) = -\frac{1}{2}\cos 2x + 3$.

$f(x) = -\frac{1}{4}\sin 2x + 3x + d$, $0 + d = 1 \Rightarrow d = 1 \Rightarrow f(x) = -\frac{1}{4}\sin 2x + 3x + 1$

4 **a** $\int_0^{\frac{\pi}{6}}\left(e^{2x-1} - \cos x\right)dx = \left[\frac{1}{2}e^{2x-1} - \sin x\right]_0^{\frac{\pi}{6}} = -0.1597\ldots \approx -0.160$

b $\int_1^2\left(e^{x\ln 2} - \frac{3}{1-2x}\right)dx = \left[\frac{1}{\ln 2}2^x + \frac{3}{2}\ln|1-2x|\right]_1^2 = \frac{4}{\ln 2} + \frac{3}{2}\ln 3 - \frac{2}{\ln 2} = 4.533 \approx 4.53$

c $\int_0^2 \frac{3x}{2x+1}dx = \int_0^2\left(\frac{3}{2} - \frac{\frac{3}{2}}{2x+1}\right)dx = \left[\frac{3}{2}x - \frac{3}{4}\ln(2x+1)\right]_0^2 = 1.792\ldots \approx 1.79$

d As $\cos 2x = \cos^2 x - \sin^2 x = (\cos x - \sin x)(\cos x + \sin x)$,

$\int_0^{\frac{\pi}{4}}\left(\frac{(\cos x - \sin x)(\cos x + \sin x)}{\cos x + \sin x}\right)dx = \int_0^{\frac{\pi}{4}}(\cos x - \sin x)dx = \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1$

5 $\frac{7x+5}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1} \therefore 7x + 5 = A(x+1) + B(2x+1)$. Letting $x = -\frac{1}{2}$ gives $A = 3$ and letting $x = -1$ gives $B = 2$. So $\frac{7x+5}{(2x+1)(x+1)} = \frac{3}{2x+1} + \frac{2}{x+1}$.

$\int\left(\frac{3}{2x+1} + \frac{2}{x+1}\right)dx = \frac{3}{2}\ln|2x+1| + 2\ln|x+1| + c$

6 $\frac{2x^2 + 2x - 8}{x^2 - 4} = 2 + \frac{2x}{x^2 - 4}$, then $\frac{2x}{x^2-4} = \frac{2x}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \therefore 2x = A(x+2) + B(x-2)$.

Letting $x = 2$ gives $A = 1$ and letting $x = -2$ gives $B = 1$. So $\frac{2x^2 + 2x - 8}{x^2 - 4} = 2 + \frac{1}{x-2} + \frac{1}{x+2}$.

$\int_0^1\left(2 + \frac{1}{x-2} + \frac{1}{x+2}\right)dx = \left[2x + \ln|x-2| + \ln|x+2|\right]_0^1 = 2 + \ln 3 - 2\ln 2$

7 a Let $u = x^2 - 7$ then $du = 2x\,dx$ and $\int 3x(x^2-7)^2 dx = \int \frac{3}{2}u^2 du = \frac{1}{2}u^3 + c = \frac{1}{2}(x^2-7)^3 + c$

b $\int 2\tan 2x\,dx = \int \frac{2\sin 2x}{\cos 2x}dx$, let $u = \cos 2x$ then $du = -2\sin 2x\,dx$ and

$\int 2\tan 2x\,dx = \int \frac{-1}{u}du = -\ln|u| + c = -\ln|\cos 2x| + c$

c Let $u = 3x - 1$ then $du = 3dx$ and $x = \frac{u+1}{3}$ then

$\int 2x\sqrt{3x-1}\,dx = \int\left(\frac{2}{9}u^{\frac{3}{2}} + \frac{2}{9}u^{\frac{1}{2}}\right)du = \frac{4}{45}u^{\frac{5}{2}} + \frac{4}{27}u^{\frac{3}{2}} + c = \frac{4}{45}(3x-1)^{\frac{5}{2}} + \frac{4}{27}(3x-1)^{\frac{3}{2}} + c$

d Let $u = x^3 - 3x^2$ then $du = (3x^2 - 6x)dx$ and

$\int (x^2-2x)\cos(x^3-3x^2)\,dx = \int \frac{1}{3}\cos u\,du = \frac{1}{3}\sin u + c = \frac{1}{3}\sin(x^3-3x^2) + c$

e Let $u = 1 - x$ then $du = -dx$ and $x = 1 - u$ then

$\int x^3\sqrt{1-x}\,dx = -\int (1-u)^3\sqrt{u}\,du = -\int(1-3u+3u^2-u^3)\sqrt{u}\,du = \int\left(u^{\frac{7}{2}} - 3u^{\frac{5}{2}} + 3u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du =$

$= \frac{2}{9}u^{\frac{9}{2}} - \frac{6}{7}u^{\frac{7}{2}} + \frac{6}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{9}(1-x)^{\frac{9}{2}} - \frac{6}{7}(1-x)^{\frac{7}{2}} + \frac{6}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$

f Let $u = x + 2$ then $du = dx$ and $x = u - 2$ then

$\int x(x+2)^7\,dx = \int (u-2)u^7\,du = \frac{1}{9}u^9 - \frac{1}{4}u^8 + c = \frac{1}{9}(x+2)^9 - \frac{1}{4}(x+2)^8 + c$

g Let $u = x - 3$ then $du = dx$ and $x = u + 3$ then

$\int x^2(x-3)^{-2}\,dx = \int (u+3)^2 u^{-2}\,du = \int(1 + 6u^{-1} + 9u^{-2})\,du =$

$= u + 6\ln|u| - 9u^{-1} + c = x - 3 + 6\ln|x-3| - \frac{9}{x-3} + c$

h Let $u = 1 + \sin 2x$ then $du = 2\cos 2x\,dx$ and

$\int \frac{\cos 2x}{(1+\sin 2x)^3}dx = \int \frac{1}{2}u^{-3}\,du = -\frac{1}{4}u^{-2} + c = -\frac{1}{4(1+\sin 2x)^2} + c$

8 a Let $u = x + 1$ then $du = dx$ and $x = u - 1$ then

$\int_1^4 \frac{x}{\sqrt{x+1}}dx = \int_2^5 (u-1)u^{-\frac{1}{2}}\,du = \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_2^5 = \frac{2}{3}\left(\sqrt{2} + 2\sqrt{5}\right) = 3.924\ldots \approx 3.92$

b Let $u = 2x + 1$ then $du = 2dx$ and $x = \frac{u-1}{2}$ then

$\int_1^3 \frac{4x}{(2x+1)^2}dx = \int_3^7 (u-1)u^{-2}\,du = \left[\ln|u| + u^{-1}\right]_3^7 = \ln 7 + \frac{1}{7} - \ln 3 - \frac{1}{3} = \ln\frac{7}{3} - \frac{4}{21} = 0.6568\ldots \approx 0.659$

c Let $u = \cos x$ then $du = -\sin x\,dx$ and

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x}dx = -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} u^{-2}\,du = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^{-2}\,du = \left[-u^{-1}\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = 2 - \frac{2}{\sqrt{3}} = 0.8452\ldots \approx 0.845$

d Let $u = 2^x$ then $du = \ln 2 \times 2^x\,dx$ and

$$\int_1^2 2^x\sqrt{2^x}\,dx = \int_2^4 \frac{1}{\ln 2}u^{\frac{1}{2}}\,du = \frac{1}{\ln 2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^4 = \frac{4\left(4-\sqrt{2}\right)}{\ln 8} = 4.974\ldots \approx 4.97$$

9 Let $u = x+1$ then $\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \int \frac{du}{u^2+1} = \arctan u + c = \arctan(x+1)+c$. So

$$\int_0^{\sqrt{3}-1} \frac{dx}{x^2+2x+2} = \left[\arctan(x+1)\right]_0^{\sqrt{3}-1} = \arctan\sqrt{3} - \arctan 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

10 a Let $u = x$ and $\frac{dv}{dx} = e^{3x}$ so $\frac{du}{dx} = 1$ and $v = \frac{1}{3}e^{3x}$ giving

$$\int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$$

b Let $u = x-1$ and $\frac{dv}{dx} = \cos x$ so $\frac{du}{dx} = 1$ and $v = \sin x$ giving

$$\int (x-1)\cos x\,dx = (x-1)\sin x - \int \sin x\,dx = (x-1)\sin x + \cos x + c$$

c Let $u = \ln 3x$ and $\frac{dv}{dx} = x^4$ so $\frac{du}{dx} = \frac{3}{3x} = \frac{1}{x}$ and $v = \frac{1}{5}x^5$ giving

$$\int x^4 \ln 3x\,dx = \frac{1}{5}x^5\ln 3x - \int \frac{1}{5}x^4\,dx = \frac{1}{5}x^5\ln 3x - \frac{1}{25}x^5 + c$$

d Let $u = \arctan x$ and $\frac{dv}{dx} = 1$ so $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = x$ giving

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) + c$$

e Let $u = \ln x$ and $\frac{dv}{dx} = x^2 - x + 3$ so $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x$ giving

$$\int \left(x^2 - x + 3\right)\ln x\,dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x\right)\ln x - \int \left(\frac{1}{3}x^2 - \frac{1}{2}x + 3\right)dx =$$

$$= \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x\right)\ln x - \frac{1}{9}x^3 - \frac{1}{4}x^2 + 3x + c$$

11 a Let $u = \arcsin x$ and $\frac{dv}{dx} = 1$ so $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $v = x$ giving

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\arcsin x + \left(1-x^2\right)^{\frac{1}{2}} + c \text{ thus}$$

$$\int_0^{\frac{1}{\sqrt{2}}} \arcsin x\,dx = \left[x\arcsin x + \left(1-x^2\right)^{\frac{1}{2}}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{4+\pi}{4\sqrt{2}} - 1 = 0.2624\ldots \approx 0.262$$

b Let $u = \ln x$ and $\frac{dv}{dx} = x^2$ so $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{3}x^3$ giving

$$\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^2\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c \text{ thus}$$

$$\int_1^e x^2\ln x\,dx = \left[\frac{1}{3}x^3\ln x - \frac{1}{9}x^3\right]_1^e = \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{1}{9}\left(1+2e^3\right) = 4.574\ldots \approx 4.57$$

c Let $u = x$ and $\frac{dv}{dx} = \cos x$ so $\frac{du}{dx} = 1$ and $v = \sin x$ giving

$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c$ thus

$$\int_0^{\frac{\pi}{3}} x \cos x \, dx = [x \sin x + \cos x]_0^{\frac{\pi}{3}} = \frac{\pi}{3} \times \frac{\sqrt{3}}{2} + \frac{1}{2} - 1 = \frac{1}{6}\left(\sqrt{3}\pi - 3\right) = 0.4069\ldots \approx 0.407$$

12 a Let $u = x^2$ and $\frac{dv}{dx} = \cos x$ so $\frac{du}{dx} = 2x$ and $v = \sin x$ giving

$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$. Using integration by parts again, let $u = 2x$ and $\frac{dv}{dx} = \sin x$ so $\frac{du}{dx} = 2$ and $v = -\cos x$ giving

$$\int x^2 \cos x \, dx = x^2 \sin x - \left(-2x \cos x + \int 2 \cos x \, dx\right) = x^2 \sin x + 2x \cos x - 2 \sin x + c$$

b Let $u = x^2 - x + 2$ and $\frac{dv}{dx} = e^{3x}$ so $\frac{du}{dx} = 2x - 1$ and $v = \frac{1}{3}e^{3x}$ giving

$\int \left(x^2 - x + 2\right)e^{3x} \, dx = \frac{1}{3}\left(x^2 - x + 2\right)e^{3x} - \frac{1}{3}\int (2x - 1)e^{3x} \, dx$. Using integration by parts again, let $u = 2x - 1$ and $\frac{dv}{dx} = e^{3x}$ so $\frac{du}{dx} = 2$ and $v = \frac{1}{3}e^{3x}$ giving

$$\int \left(x^2 - x + 2\right)e^{3x} \, dx = \frac{1}{3}\left(x^2 - x + 2\right)e^{3x} - \frac{1}{3}\left(\frac{1}{3}(2x - 1)e^{3x} - \int \frac{2}{3}e^{3x} \, dx\right) =$$

$$= \frac{1}{3}\left(x^2 - x + 2\right)e^{3x} - \frac{1}{9}(2x - 1)e^{3x} + \frac{2}{27}e^{3x} + c = \frac{e^{3x}}{27}\left(9x^2 - 15x + 23\right) + c$$

13 a Let $u = x^2 - 1$ and $\frac{dv}{dx} = e^{2x}$ so $\frac{du}{dx} = 2x$ and $v = \frac{1}{2}e^{2x}$ giving

$\int \left(x^2 - 1\right)e^{2x} \, dx = \frac{1}{2}\left(x^2 - 1\right)e^{2x} - \frac{1}{2}\int 2xe^{2x} \, dx$. Using integration by parts again, let $u = 2x$ and $\frac{dv}{dx} = e^x$ so $\frac{du}{dx} = 2$ and $v = \frac{1}{2}e^{2x}$, giving

$$\int \left(x^2 - 1\right)e^{2x} \, dx = \frac{1}{2}\left(x^2 - 1\right)e^{2x} - \frac{1}{2}\left(xe^{2x} - \int e^{2x} \, dx\right) = \frac{1}{2}\left(x^2 - 1\right)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c \text{ thus}$$

So $\int_0^3 \left(x^2 - 1\right)e^{2x} \, dx = \left[\frac{1}{2}\left(x^2 - 1\right)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x}\right]_0^3 = \frac{1}{4}\left(1 + 11e^6\right) \approx 1109.68$

b Let $u = x^3$ and $\frac{dv}{dx} = \cos x$ so $\frac{du}{dx} = 3x^2$ and $v = \sin x$ giving

$\int x^3 \cos x \, dx = x^3 \sin x - 3\int x^2 \sin x \, dx$. Using integration by parts again, let $u = x^2$ and $\frac{dv}{dx} = \sin x$ so $\frac{du}{dx} = 2x$ and $v = -\cos x$ giving

$\int x^3 \cos x \, dx = x^3 \sin x - 3\left(-x^2 \cos x + \int 2x \cos x \, dx\right) = x^3 \sin x + 3x^2 \cos x - 6\int x \cos x \, dx$. Using integration by parts again, let $u = x$ and $\frac{dv}{dx} = \cos x$ so $\frac{du}{dx} = 1$ and $v = \sin x$ giving

$\int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - 6\left(x \sin x - \int \sin x \, dx\right) = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$ thus

$$\int_0^{\frac{\pi}{3}} x^3 \cos x \, dx = \left[x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x\right]_0^{\frac{\pi}{3}} = 3 - \sqrt{3}\pi + \frac{\pi^2}{6} + \frac{\pi^3}{18\sqrt{3}} = 0.1980\ldots \approx 0.198$$

c Let $u = x^2$ and $\frac{dv}{dx} = e^{-x}$ so $\frac{du}{dx} = 2x$ and $v = -e^{-x}$ giving $\int x^2 e^{-x}\,dx = -x^2 e^{-x} + \int 2xe^{-x}\,dx$.

Using integration by parts again, let $u = 2x$ and $\frac{dv}{dx} = -e^{-x}$ so $\frac{du}{dx} = 2$ and $v = -e^{-x}$ giving

$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + \left(-2xe^{-x} + \int 2e^{-x}\,dx\right) = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$ thus

$\int_0^2 \frac{x^2}{e^x} dx = \left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^2 = 2 - 10e^{-2} = 0.6464\ldots \approx 0.646$

14 Set $I = \int e^x \sin 3x\,dx$ then let $u = e^x$ and $\frac{dv}{dx} = \sin 3x$ so $\frac{du}{dx} = e^x$ and $v = -\frac{1}{3}\cos 3x$ then

$I = -\frac{1}{3}e^x \cos 3x + \frac{1}{3}\int e^x \cos 3x\,dx$. Using integration by parts again, let $u = e^x$ and $\frac{dv}{dx} = \cos 3x$

so $\frac{du}{dx} = e^x$ and $v = \frac{1}{3}\sin 3x$ thus giving

$I = -\frac{1}{3}e^x \cos 3x + \frac{1}{9}e^x \sin 3x - \frac{1}{9}\int e^x \sin 3x\,dx = -\frac{1}{3}e^x \cos 3x + \frac{1}{9}e^x \sin 3x - \frac{1}{9}I$ so

$I = \frac{1}{10}\left(e^x \sin 3x - 3e^x \cos 3x\right) + c$.

8.1 Areas and volumes

1 **Do not use a GDC for this question**

Find the area of the region enclosed by the graphs of:

a $y = 2x^3 - x^2 + 3x + 1$, $y = x^3 + x^2 + 4x - 1$

b $y = x^4 - 10$, $y = 10 - x^2$

2 Find the area of the region enclosed by the graphs of:

a $y = e^{x^2}$, $y = 4\cos x$

b $y = x^4 - 1$, $y = \dfrac{2}{1+x}$

3 Find the area bounded by the graphs of $y = 3^x$, $y = 5^x$ and $y = \dfrac{1}{x^3}$

4 **Do not use a GDC for this question**

Find the volume of the solid formed by rotating the region enclosed by the graph of the function and the x-axis through 2π radians about the x-axis in the given interval:

a $y = x^2 + 2x$, $[0, 2]$

b $y = \cos x$, $\left[0, \dfrac{\pi}{2}\right]$

5 **Do not use a calculator for this question**

A glass is modelled by the function $y = 7x^{\frac{3}{2}} + 1$, $x > 0$ rotated 2π radians about the y-axis between $y = 1$ and $y = 8$. Find the volume of the glass.

6 Find the volume of the solid generated by rotating the region in the first quadrant bounded by the curves $y = \dfrac{1}{4}e^x$ and $y = \sin x$ through 2π radians about the x-axis.

Answers

1 **a** Graphs cross when $2x^3 - x^2 + 3x + 1 = x^3 + x^2 + 4x - 1 \Rightarrow x^3 - 2x^2 - x + 2 = 0$ which occurs at $(x-1)(x+1)(x-2) = 0$, i.e. $x = -1, 1, 2$. By considering the shape of the graphs, area is calculated as

$$\int_{-1}^{1}\left(x^3 - 2x^2 - x + 2\right)dx + \int_{1}^{2}\left(-x^3 + 2x^2 + x - 2\right)dx = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right]_{-1}^{1} + \left[-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} - 2x\right]_{1}^{2}$$

$$= \left(\frac{13}{12} + \frac{19}{12}\right) + \left(-\frac{2}{3} + \frac{13}{12}\right)$$

$$= \frac{37}{12}$$

b Graphs cross when $x^4 - 10 = 10 - x^2 \Rightarrow x^4 + x^2 - 20 = 0 \Rightarrow \left(x^2 + 5\right)\left(x^2 - 4\right) = 0$. By considering the shape of the graphs, area is $\int_{-2}^{2}\left(20 - x^2 - x^4\right)dx = \left[20x - \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-2}^{2} = \frac{464}{15} + \frac{464}{15} = \frac{928}{15}$

2 **a** From GDC, $x = \pm 0.932...$ so area $= \int_{-0.932...}^{0.932...}\left(4\cos x - e^{x^2}\right)dx = 3.843... \approx 3.84$

b From GDC, $x = 1.072...$ so area $= \int_{0}^{1.072...}\left(\frac{1}{2+x} - \left(x^4 - 1\right)\right)dx = 1.218... \approx 1.22$

3 From GDC, $5^x = \frac{1}{x^3} \Rightarrow x = 0.6904...$ and $3^x = \frac{1}{x^3} \Rightarrow x = 0.7576...$ so area

$$= \int_{0}^{0.6904...}(5^x - 3^x)\,dx + \int_{0.6904...}^{0.7597...}\left(\frac{1}{x^3} - 3^x\right)dx = 0.2620... \approx 0.262$$

4 **a** Volume $= \pi\int_{0}^{2}\left(x^2 + 2x\right)^2 dx = \pi\int_{0}^{2}\left(x^4 + 4x^3 + 4x^2\right)dx = \pi\left[\frac{1}{5}x^5 + x^4 + \frac{4}{3}x^3\right]_{0}^{2} = \frac{496\pi}{15}$

b Volume $= \pi\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx = \pi\int_{0}^{\frac{\pi}{2}}\left(\frac{1}{2}(\cos 2x + 1)\right)dx = \pi\left[\frac{1}{4}\sin 2x + \frac{1}{2}x\right]_{0}^{\frac{\pi}{2}} = \frac{\pi^2}{4}$

5 $y = 7x^{\frac{3}{2}} + 1 \Rightarrow x = \left(\frac{y-1}{7}\right)^{\frac{2}{3}}$, so volume $= \pi\int_{1}^{8}\left(\frac{y-1}{7}\right)^{\frac{4}{3}}dy$. Let $u = \frac{y-1}{7}$ then volume

$$= \pi\int_{0}^{1}7u^{\frac{4}{3}}du = \pi\left[3u^{\frac{7}{3}}\right]_{0}^{1} = 3\pi$$

6 From GDC, $\frac{1}{4}e^x = \sin x \Rightarrow x = 0.3705..., 1.364...$ so volume

$$= \pi\int_{0.3705...}^{1.364...}\left(\sin x - \frac{1}{4}e^x\right)^2 dx = 0.04686... \approx 0.0469$$

8.2 Kinematics

1 A particle travels with velocity $v(t) = (1 - 2t)e^t$, $0 \le t \le 1$ where v is in metres per second and t is in seconds. If the displacement is 0 when $t = 0$

a Find the particle's displacement after 1 second.

b Find the total distance travelled in the second.

c Check your answer to **b** using technology.

2 **Do not use a calculator for this question**

Given the velocity function $v(t) = \sqrt{3 - t}$, $0 \le t \le 3$ find

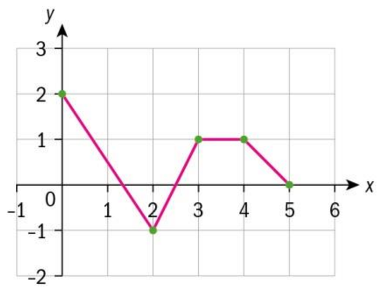
a The times when the particle

i stops **ii** moves to the left **iii** moves to the right

b The particle's displacement at the end of the time interval.

c The total distance travelled.

3 The graph shows the velocity in metres per second of a particle moving along the x-axis. At $t = 0$, the particle is 3cm to the right of the origin



a Find the particle's position at the end of the 5 seconds.

b Find the total distance travelled.

Answers

1 **a** $\int(1-2t)e^t\,dt$, use integration by parts: let $u=1-2t$ and $\frac{dv}{dt}=e^t$ then $\frac{du}{dt}=-2$ and $v=e^t$

giving $\int(1-2t)e^t\,dt=(1-2t)e^t+2e^t+c=(3-2t)e^t+c$.

$3e^0+c=0\Rightarrow c=-3\Rightarrow s=(3-2t)e^t-3$ so when $t=1$, $s=e-3$

b $v=0$ when $t=\frac{1}{2}$ so $s=2e^{\frac{1}{2}}-3$ giving distance $=2e^{\frac{1}{2}}-3+2e^{\frac{1}{2}}-3-(e-3)=4e^{\frac{1}{2}}-e-3$

2 **a** **i** $t=3$ **ii** $0\le t<3$ **iii** Never

b $\int_0^3(3-t)^{\frac{1}{2}}\,dt=\left[-\frac{2}{3}(3-t)^{\frac{3}{2}}\right]_0^3=\frac{2}{3}\times3^{\frac{3}{2}}=2\sqrt{3}$

c $2\sqrt{3}$ as the velocity never changes direction

3 **a** Area $=\frac{1}{2}\times\frac{2}{3}\times2-\frac{1}{2}\times\frac{1}{3}\times1-\frac{1}{2}\times\left(2\frac{1}{2}+1\right)\times1=\frac{2}{3}-\frac{1}{6}+\frac{7}{4}=\frac{9}{4}$, so position $=\frac{9}{4}+3=\frac{21}{4}$

b Distance travelled $=\frac{2}{3}+\frac{1}{6}+\frac{7}{4}=\frac{31}{12}$

8.3 Ordinary differential equations (ODEs)

1 Solve each differential equation:

a $\frac{dy}{dx} = \frac{2x^3}{3y^2}$ **b** $\frac{dy}{dx} = y \sin x$ **c** $\frac{dy}{dx} = \frac{x^2 - \cos x}{3y^3}$

2 Solve each differential equation given the initial value:

a $x^2 + y^2 \frac{dy}{dx} = 0, y(1) = 2$ **b** $\frac{dy}{dx} = 2xy \sec^2 x, y(0) = 1$

3 Newton's Law of Cooling states that the rate at which an object's temperature is changing at any given time is proportional to the difference between its temperature and the temperature of the surrounding environment.

An object's temperature is $90^\circ C$ at 10:00. At 10:30 its temperature is $73^\circ C$. If the surrounding temperature is a constant $25^\circ C$,

a Formulate a differential equation to model this situation

b Find the temperature of the object at 11:00

4 A colony of bacteria grows in population P at a rate proportional to the size of the population. After 2 hours the population is 700 and after 3 hours it has increased to 800.

a Formulate a differential equation to model this situation

b Solve the differential equation to find an expression for P

c Find what the initial population was

d How many hours will it take for the population to reach 2000

5 The rate of spread of a disease can be modelled by the logistic equation $\frac{dI}{dt} = 0.6I\left(1 - \frac{I}{350}\right)$ where I is the number of people infected. Initially $I = 20$.

a Solve the differential equation **b** Find the value of I when $t = 5$

6 Solve these differential equations using the substitution $v = \frac{y}{x}$:

a $xy' = x^2e^x + y$ **b** $x^2y' = y^2 + xy + x^2$ **c** $y' = \frac{y}{x} + \frac{2y^2}{x^2}$, $y(1) = 4$

7 Find the general solution of these differential equations:

a $y' - y = \sin x$ **b** $xy' + y = x^3 + x$ **c** $x^2y' + 2xy = \sin 2x$

8 Solve these initial value problems:

a $xy' = y - x^2 \cos x$, $y\left(\frac{\pi}{2}\right) = 0$ **b** $\frac{dy}{dx} + y \cot x = \operatorname{cosec} x$, $y\left(\frac{\pi}{2}\right) = 3$

9 Use Euler's method to find approximate solutions to these differential equations at the stated value, given the initial value and the step length.

a $y' + 4y = 2 - e^{4x^2}$, $y(0) = 1$, at $x = 0.4$, step length $= 0.1$

b $y' = e^{2x} - y^3$, $y(0) = 3$, at $x = 0.25$, step length $= 0.05$

Answers

1 **a** $\int 3y^2\,dy = \int 2x^3\,dx \Rightarrow y^3 = \frac{x^4}{2} + c \Rightarrow y = \sqrt[3]{\frac{x^4}{2} + c}$

b $\int \frac{1}{y}\,dy = \int \sin x\,dx \Rightarrow \ln y = -\cos x + c \Rightarrow y = e^{-\cos x + c} = Ae^{-\cos x}$

c $\int 3y^3\,dy = \int \left(x^2 - \cos x\right)dx \Rightarrow \frac{3y^4}{4} = \frac{x^3}{3} - \sin x + c \Rightarrow y = \sqrt[4]{\frac{4x^3}{3} - \frac{4}{3}\sin x + c}$

2 **a** $\int y^2\,dy = \int -x^2\,dx \Rightarrow \frac{y^3}{3} = -\frac{x^3}{3} + c$, using $y(1) = 2$, $\frac{8}{3} = -\frac{1}{3} + c \Rightarrow c = 3 \Rightarrow x^3 + y^3 = 9$

b $\int \frac{1}{y}\,dy = \int 2x\sec^2 x\,dx$ using integration by parts, let $u = 2x$ and $\frac{dv}{dx} = \sec^2 x$ then $\frac{du}{dx} = 2$ and $v = \tan x$ giving $\int 2x\sec^2 x\,dx = 2x\tan x - \int 2\tan x\,dx = 2x\tan x + 2\ln|\cos x|$ so $\ln y = 2x\tan x + 2\ln|\cos x| + c \Rightarrow y = Ae^{2x\tan x}\cos^2 x$ using $y(0) = 1$,
$1 = Ae^0\cos^2 0 \Rightarrow A = 1 \Rightarrow y = e^{2x\tan x}\cos^2 x$

3 **a** $\frac{dT}{dt} = -k\left(T - 25\right)$

b $\int \frac{dT}{T - 25} = -kt + c \Rightarrow \ln\left(T - 25\right) = -kt + c \Rightarrow T = 25 + Ae^{-kt}$. Let t be the number of hours after 10:00, then $90 = 25 + Ae^0 \Rightarrow A = 65 \Rightarrow T = 25 + 65e^{-kt}$ and
$73 = 25 + 65e^{-k\times 0.5} \Rightarrow k = 2\ln\frac{65}{48} \Rightarrow T = 25 + 65e^{-2t\ln\frac{65}{48}}$ so when t = 1,
$T = \frac{3929}{65} = 60.446\ldots \approx 60.4^\circ C$

4 **a** $\frac{dP}{dt} = kP$

b $P = Ae^{kt}$ using $Ae^{2k} = 700$ and $Ae^{3k} = 800$ we get $e^k = \frac{8}{7} \Rightarrow k = \ln\frac{8}{7} = 0.1335\ldots$ and
$A = \frac{8575}{16} = 535.9\ldots$ so $P = 536e^{0.134t}$

c 536

d $536e^{0.134t} = 2000 \Rightarrow t = \frac{1}{0.134}\ln\left(\frac{2000}{536}\right) = 9.861\ldots \approx 9.86$ hours

5 **a** $\int \frac{dI}{I(350 - I)} = 0.6t + c \Rightarrow \int\left(\frac{1}{I} - \frac{1}{350 - I}\right)dI = 0.6t + c \Rightarrow \ln\left(\frac{I}{350 - I}\right) = 0.6t + c \Rightarrow \frac{I}{350 - I} = Ae^{0.6t}$

Using the initial condition, $Ae^{0.6\times 0} = A = \frac{20}{350 - 20} = \frac{2}{33}$ so

$$\frac{I}{350 - I} = \frac{2}{33}e^{0.6t} \Rightarrow I = \frac{\frac{700}{33}e^{0.6t}}{1 + \frac{2}{33}e^{0.6t}} = \frac{700e^{0.6t}}{33 + 2e^{0.6t}}$$

b $I = \frac{700e^3}{33 + 2e^3} = 192.1 \approx 192$

6 a $v + xv' = xe^x + v \Rightarrow v' = e^x \Rightarrow v = e^x + c \Rightarrow y = xe^x + cx$

b $x^2(v + xv') = v^2x^2 + vx^2 + x^2 \Rightarrow v + xv' = v^2 + v + 1 \Rightarrow \int \frac{dv}{v^2+1} = \int \frac{1}{x}dx$

$\Rightarrow \arctan v = \ln x + c \Rightarrow y = x\tan(\ln x + c)$

c $v + xv' = v + 2v^2 \Rightarrow \int \frac{dv}{2v^2} = \frac{dx}{x} \Rightarrow -\frac{1}{v} = \ln x + c \Rightarrow v = -\frac{1}{\ln x + c}$

$\Rightarrow y = -\frac{x}{\ln x + c}\ 4 = -\frac{1}{c} \Rightarrow c = -\frac{1}{4} \Rightarrow y = -\frac{4x}{1-4\ln x}$

7 a Integrating factor $= e^{-x}$, so $\frac{d}{dx}\left(e^{-x}y\right) = e^{-x}\sin x \Rightarrow e^{-x}y = \int e^{-x}\sin x\,dx$. Let $u = e^{-x}$ and $\frac{dv}{dx} = \sin x$ then $\frac{du}{dx} = -e^{-x}$ and $v = -\cos x$ so $\int e^{x}\sin x\,dx = e^{-x}\cos x - \int e^{-x}\cos x\,dx$. Using integration by parts again, let $u = e^{-x}$ and $\frac{dv}{dx} = \cos x$ then $\frac{du}{dx} = -e^{-x}$ and $v = \sin x$ so $\int e^{-x}\sin x\,dx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx \Rightarrow 2\int e^{-x}\sin x\,dx = -e^{-x}\cos x - e^{-x}\sin x$ solving this gives $\int e^{-x}\sin x\,dx = \frac{1}{2}\left(-e^{-x}\sin x - e^{-x}\cos x\right) + c$ and so the solution is

$y = \frac{1}{2}\left(-\sin x - \cos x\right) + ce^{x}$

b $y' + \frac{y}{x} = x^2 + 1$. Integrating factor $= e^{\int \frac{1}{x}dx} = e^{\ln x} = x$, giving

$\frac{d}{dx}(xy) = x^3 + x \Rightarrow xy = \frac{1}{4}x^4 + \frac{1}{2}x^2 + c \Rightarrow y = \frac{1}{4}x^3 + \frac{1}{2}x + \frac{c}{x}$

c $y' + \frac{2y}{x} = \frac{\sin 2x}{x^2}$. Integrating factor $= e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$, giving

$\frac{d}{dx}\left(x^2y\right) = \sin 2x \Rightarrow x^2y = -\frac{1}{2}\cos 2x + c \Rightarrow y = \frac{c - \cos 2x}{2x^2}$

8 a $y' - \frac{1}{x}y = -x\cos x$. Integrating factor $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$, giving

$\frac{d}{dx}\left(\frac{1}{x}y\right) = -\cos x \Rightarrow \frac{1}{x}y = -\sin x + c \Rightarrow y = cx - x\sin x$, using the initial value,

$0 = c\frac{\pi}{2} - \frac{\pi}{2} \Rightarrow c = 1 \Rightarrow y = x - x\sin x$

b Integrating factor $= e^{\int \cot x\,dx} = e^{\ln \sin x} = \sin x$, giving $\frac{d}{dx}(\sin xy) = 1 \Rightarrow \sin xy = x + c \Rightarrow y = \frac{x+c}{\sin x}$

Using the initial value, $3 = \frac{\frac{\pi}{2} + c}{1} \Rightarrow c = 3 - \frac{\pi}{2} \Rightarrow y = \frac{x + 3 - \frac{\pi}{2}}{\sin x}$

9 a

n	x_n	y_n	$\frac{dy}{dx} = 2 - e^{4x_n^2} - 4y_n$
0	0	1	-3
1	0.1	$1 - 3 \times 0.1 = 0.7$	$-1.840\ldots$
2	0.2	$0.7 - 1.840\ldots \times 0.1 = 0.5159\ldots$	$-1.237\ldots$
3	0.3	$0.5159\ldots - 1.237\ldots \times 0.1 = 0.3922\ldots$	$-1.002\ldots$
4	0.4	$0.3922\ldots - 1.002\ldots \times 0.1 = 0.2919\ldots$	

$y(0.4) \approx 0.292$

b

n	x_n	y_n	$\frac{dy}{dx} = e^{2x_n} - {y_n}^3$
0	0	3	-26
1	0.05	$3 - 26 \times 0.05 = 1.7$	$-3.807\ldots$
2	0.1	$1.7 - 3.807\ldots \times 0.05 = 1.509\ldots$	$-2.218\ldots$
3	0.15	$1.509\ldots - 2.218\ldots \times 0.05 = 1.398\ldots$	$-1.386\ldots$
4	0.2	$1.398\ldots - 1.386\ldots \times 0.05 = 1.329\ldots$	$-0.8573\ldots$
5	0.25	$1.329\ldots - 0.8573\ldots \times 0.05 = 1.286\ldots$	

$y(0.25) \approx 1.29$

8.4 Limits revisited

1 Use L'Hopital's Rule to find the following limits if possible:

a $\lim_{x\to 0}\frac{\sin 3x}{x}$ **b** $\lim_{x\to \frac{e}{2}}\frac{1-\ln 2x}{\frac{2x}{e}-1}$ **c** $\lim_{x\to \sqrt{3}}\frac{\arctan(x)-\frac{\pi}{3}}{\tan\left(\frac{\pi x}{3\sqrt{3}}\right)-\sqrt{3}}$

d $\lim_{x\to \infty}\frac{3x^2}{e^{2-3x}}$ **e** $\lim_{x\to 0}\frac{\cos 2x-\cos 5x}{x^2}$

2 Use L'Hopital's Rule, if possible, to find any asymptotes of these functions:

a $y=\frac{4x^3-3x^2+2x-1}{1-x-x^3}$ **b** $y=\frac{3^x}{x^3}$

3 Compute the Maclaurin polynomials to approximate the following functions around $x=0$ to the given order:

a $f(x)=x^2\sin x$, order 5 **b** $f(x)=\cos(-2x)$, order 4

c $f(x)=e^{2x^2}$, order 6 **d** $f(x)=\frac{1}{1+2x}$, order 4

4 Use an appropriate known Maclaurin or Binomial series to find the Maclaurin expansions for:

a $y=\frac{1}{1-x^2}$ **b** $y=e^{4x}$ **c** $y=\cos^2 x$ (use $\cos 2x=2\cos^2 x-1$)

5 Use partial fractions to find a Maclaurin series for $f(x)=\frac{5x+3}{x^2+2x-3}$

6 Find the following limits using Maclaurin series:

a $\lim_{x\to 0}\frac{e^{x^2}-e^{-2x^2}}{x^2}$ **b** $\lim_{x\to 0}\frac{x^2-\cos^2 x+1}{x^2\cos^2 x}$

7 Use the Binomial Series to find a Maclaurin expansion for:

a $f(x)=\frac{1}{\sqrt{1-x}}$ **b** $f(x)=\frac{1}{\sqrt[3]{1+x^2}}$

8 **a** Find the Maclaurin expansion for $f(x)=\sqrt{1+x}$

b Show that $\sqrt{50}=7\sqrt{1+\frac{1}{49}}$

c Use your answer to a to find $\sqrt{50}$ correct to 5 decimal places.

9 Find the first 3 terms of a Maclaurin expansion to approximate the solution of $y' - ye^{2x} = e^{x}$ if $y(0) = 1$

Answers

1 **a** $\lim_{x\to 0}\frac{\sin 3x}{x}=\lim_{x\to 0}\frac{3\cos 3x}{1}=3$ **b** $\lim_{x\to \frac{e}{2}}\frac{1-\ln 2x}{\frac{2x}{e}-1}=\lim_{x\to \frac{e}{2}}\frac{-\frac{2}{2x}}{\frac{2}{e}}=-1$

c $\lim_{x\to\sqrt{3}}\frac{\arctan(x)-\frac{\pi}{3}}{\tan\left(\frac{\pi x}{3\sqrt{3}}\right)-\sqrt{3}}=\lim_{x\to\sqrt{3}}\frac{\frac{1}{1+x^2}}{\frac{\pi}{3\sqrt{3}}\sec^2\left(\frac{\pi x}{3\sqrt{3}}\right)}=\frac{\frac{1}{4}}{\frac{\pi}{3\sqrt{3}}\times 4}=\frac{3\sqrt{3}}{16\pi}$

d $\lim_{x\to\infty}\frac{3x^2}{e^{2-3x}}=\lim_{x\to\infty}\frac{6x}{-3e^{2-3x}}=\lim_{x\to\infty}\frac{6}{-9e^{2-3x}}$, which tends to ∞ as $x\to\infty$ so no finite limit

e $\lim_{x\to 0}\frac{\cos 2x-\cos 5x}{x^2}=\lim_{x\to 0}\frac{-2\sin 2x+5\sin 5x}{2x}=\lim_{x\to 0}\frac{-4\cos 2x+25\cos 5x}{2}=\frac{21}{2}$

2 **a** $\lim_{x\to\infty}\frac{12x^2-6x+2}{-1-3x^2}=\lim_{x\to\infty}\frac{24x-6}{-6x}=\lim_{x\to\infty}\frac{24}{-6}=-4$ so $y=-4$ is a horizontal asymptote

b $\lim_{x\to\infty}\frac{\ln 3\times 3^x}{3x^2}=\lim_{x\to\infty}\frac{(\ln 3)^2\times 3^x}{6x}=\lim_{x\to\infty}\frac{(\ln 3)^3\times 3^x}{6}$ which tends to ∞ as $x\to\infty$ so no horizontal asymptotes

3 **a** $f(x)=\sin x\Rightarrow f(0)=0$, $f'(x)=\cos x\Rightarrow f'(0)=1$, $f''(x)=-\sin x\Rightarrow f''(0)=0$,

$f'''(x)=-\cos x\Rightarrow f'''(0)=-1$ so $\sin x=x-\frac{x^3}{6}+\frac{x^5}{120}-\ldots\Rightarrow x^2\sin x\approx x^3-\frac{x^5}{6}$

b $f(x)=\cos x\Rightarrow f(0)=1$, $f'(x)=-\sin x\Rightarrow f'(0)=0$, $f''(x)=-\cos x\Rightarrow f''(0)=-1$,

$f'''(x)=\sin x\Rightarrow f'''(0)=0$ so $\cos x=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\ldots\Rightarrow\cos(-2x)\approx 1-2x^2+\frac{2x^4}{3}$

c $f(x)=e^x\Rightarrow f(0)=1$, $f'(x)=e^x\Rightarrow f'(0)=1$, $f''(x)=e^x\Rightarrow f''(0)=1$, $f'''(x)=e^x\Rightarrow f'''(0)=1$ so

$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\Rightarrow e^{2x^2}\approx 1+2x^2+2x^4+\frac{4x^6}{3}$

d $f(x)=(1+2x)^{-1}\Rightarrow f(0)=1$, $f'(x)=-2(1+2x)^{-2}\Rightarrow f'(0)=-2$, $f''(x)=4(1+2x)^{-3}\Rightarrow f''(0)=4$,

$f'''(x)=-12(1+2x)^{-4}\Rightarrow f'''(0)=-12$ $f^{(4)}(x)=48(1+2x)^{-4}\Rightarrow f^{(4)}(0)=48$ so

$(1+2x)^{-1}\approx 1-2x+4x^2-8x^3+16x^4$

4 **a** $\sum_{n=0}^{\infty}\frac{(-1)(-2)\ldots..(-1-n+1)}{n!}\left(-x^2\right)^n=\sum_{n=0}^{\infty}x^{2n}$ **b** $\sum_{n=0}^{\infty}\frac{(4x)^n}{n!}=\sum_{n=0}^{\infty}\frac{4^n}{n!}x^n$

c $\frac{1}{2}(1+\cos 2x)=\frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}=\frac{1}{2}+\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n-1}}{(2n)!}x^{2n}$

5 $\frac{5x+3}{x^2+2x-3}=\frac{A}{x-1}+\frac{B}{x+3}\Rightarrow 5x+3=A(x+3)+B(x-1)$, let $x=1$ to find $A=2$ and let $x=-3$ to find $B=3$. So $\frac{5x+3}{x^2+2x-3}=\frac{2}{x-1}+\frac{3}{x+3}=(-2)(1-x)^{-1}+\left(1+\frac{x}{3}\right)^{-1}$. So

$$\frac{5x+3}{x^2+2x-3}=\sum_{n=0}^{\infty}\left((-2)\frac{(-1)(-2)\ldots.(-1-n+1)}{n!}(-x)^n+\frac{(-1)(-2)\ldots.(-1-n+1)}{n!}\left(\frac{x}{3}\right)^n\right)$$

$$=\sum_{n=0}^{\infty}(-2)x^n+(-1)^n\left(\frac{x}{3}\right)^n=\sum_{n=0}^{\infty}\left(-2+\left(-\frac{1}{3}\right)^n\right)x^n$$

6 a

$$\lim_{x\to 0}\frac{e^{x^2}-e^{-2x^2}}{x^2}=\lim_{x\to 0}\frac{\left(1+x^2+\frac{x^4}{2}+\ldots\right)-\left(1-2x^2+2x^4+\ldots\right)}{x^2}=\lim_{x\to 0}\frac{3x^2-\frac{3}{2}x^4+\ldots}{x^2}=\lim_{x\to 0}\frac{3-\frac{3}{2}x^2+\ldots}{1}=3$$

b From **4c** $\cos^2 x=1-x^2+\ldots$ so $\lim_{x\to 0}\frac{x^2-\cos^2 x+1}{x^2\cos^2 x}=\lim_{x\to 0}\frac{x^2-\left(1-x^2+\ldots\right)+1}{x^2\left(1-x^2+\ldots\right)}=\lim_{x\to 0}\frac{2x^2+\ldots}{x^2+\ldots}=2$

7 a $(1-x)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots..\left(-\frac{1}{2}-n+1\right)}{n!}(-1)^n x^n$

b $\left(1+x^2\right)^{-\frac{1}{3}}=\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\ldots..\left(-\frac{1}{3}-n+1\right)}{n!}x^{2n}$

8 a $(1+x)^{\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\ldots..\left(\frac{1}{2}-n+1\right)}{n!}x^n$

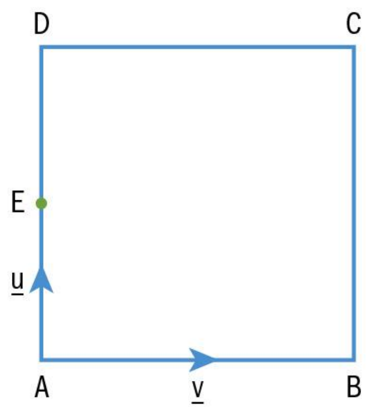
b $\sqrt{50}=\sqrt{49+1}=7\sqrt{1+\frac{1}{49}}$

c From part **a**, $\sqrt{50}=7\left(1+\frac{1}{2}\times\frac{1}{49}-\frac{1}{8}\times\left(\frac{1}{49}\right)^2+\frac{1}{16}\times\left(\frac{1}{49}\right)^3-\frac{5}{128}\times\left(\frac{1}{49}\right)^4+\ldots\right)\approx 7.07107$

9 $y(0)=1$, $y'-ye^{2x}=e^x\Rightarrow y'=ye^{2x}+e^x\Rightarrow y'(0)=2$, $y''=2ye^{2x}+y'e^{2x}+e^x\Rightarrow y''(0)=5$ so $y(x)\approx 1+2x+\frac{5x^2}{2}$

9.1 Geometrical representation of vectors and basic operations

1 ABCD is a square and E is the midpoint of AD.



Letting $\overrightarrow{\mathbf{AD}} = \mathbf{u}$ and $\overrightarrow{\mathbf{AB}} = \mathbf{v}$, write the following in terms of $\mathbf{u}$ and $\mathbf{v}$:

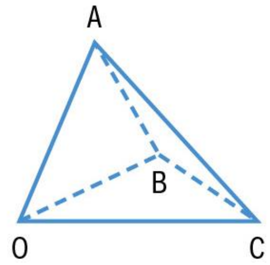
a $\overrightarrow{\mathbf{AC}}$

b $\overrightarrow{\mathbf{DB}}$

c $\overrightarrow{\mathbf{CE}}$

2 Show that the diagonals of a parallelogram bisect each other.

3 OABC is a tetrahedron (triangle based pyramid) as shown in the diagram.



The centroid of the tetrahedron is defined to be the point X such that

$$\overrightarrow{\mathbf{OX}} = \frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$$

Show that AX meets the base OBC at the point Y such that $\overrightarrow{\mathbf{OY}} = \frac{1}{3}(\mathbf{b}+\mathbf{c})$

Answers

1 **a** $\mathbf{u}+\mathbf{v}$

b $\mathbf{v}-\mathbf{u}$

c $-\mathbf{v}-\frac{1}{2}\mathbf{u}$

2 Let parallelogram be *ABCD* so $\overrightarrow{AB}=\overrightarrow{DC}$ and $\overrightarrow{AD}=\overrightarrow{BC}$

Let *E* be the midpoint of diagonal *AC*

$$\Rightarrow \overrightarrow{\mathbf{AE}}=\frac{1}{2}\left(\overrightarrow{\mathbf{AB}}+\overrightarrow{\mathbf{BC}}\right)$$

Let *F* be the midpoint of diagonal *BD*

$$\Rightarrow \overrightarrow{\mathbf{AE}}=\frac{1}{2}\left(\overrightarrow{\mathbf{AB}}+\frac{1}{2}\left(\overrightarrow{\mathbf{BC}}+\overrightarrow{\mathbf{CD}}\right)\right)$$

$$=\frac{1}{2}\left(\overrightarrow{\mathbf{AB}}+\frac{1}{2}\left(\overrightarrow{\mathbf{BC}}-\overrightarrow{\mathbf{AB}}\right)\right)$$

$$\frac{1}{2}\left(\overrightarrow{\mathbf{AB}}+\overrightarrow{\mathbf{BC}}\right)$$

So proved.

3

$$\overrightarrow{\mathbf{AX}}=\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{a}=\frac{1}{4}\mathbf{b}+\frac{1}{4}\mathbf{c}-\frac{3}{4}\mathbf{a}$$

$$\overrightarrow{\mathbf{AY}}=\frac{1}{3}\mathbf{b}+\frac{1}{3}\mathbf{c}-\mathbf{a}=\frac{4}{3}\overrightarrow{\mathbf{AX}}$$

So *Y* lies on *AX.*

9.2 Introduction to vector algebra

1 Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j},$ find

a $2\mathbf{a} - \mathbf{b}$

b $\frac{1}{3}\mathbf{a} + \frac{3}{4}\mathbf{b}$

2 If express the following as linear combinations of $\mathbf{p}$ and $\mathbf{q}$:

a $\mathbf{i} + 3\mathbf{j}$

b $5\mathbf{i} - 7\mathbf{j}$

3 *A*, *B* and *C* have coordinates (1,2,1), (3,4,-2) and (2,7,-4) respectively. If *ABCD* is a parallelogram, find the coordinates of *D*.

4 Find a unit vector in the direction of these vectors:

a $8\mathbf{i} - 15\mathbf{j}$

b $3\mathbf{i} - \mathbf{j} - 7\mathbf{k}$

5 Find the vectors of length 3 parallel to:

a $4\mathbf{i} - 5\mathbf{j}$

b $3\mathbf{i} - 4\mathbf{j} - 12\mathbf{k}$

6 *A* is the point (1,2,4), *B* is the point (2,3,7) and *C* is the point (4,-1,4). Find the lengths of *AB, AC* and *BC.*

Hence show that $\angle BAC = 90^\circ$.

Answers

1 **a** $2\mathbf{i} - 7\mathbf{j}$

b $4\mathbf{i} + \frac{19}{12}\mathbf{j}$

2 **a** $\mathbf{i} + 3\mathbf{j} = a\mathbf{p} + b\mathbf{q}$

$$\Rightarrow 2a + 3b = 1, a - b = 3$$
$$\Rightarrow a = 2, b = -1$$
$$\Rightarrow \mathbf{i} + 3\mathbf{j} = 2\mathbf{p} - \mathbf{q}$$

b $5\mathbf{i} - 7\mathbf{j} = a\mathbf{p} + b\mathbf{q}$

$$\Rightarrow 2a + 3b = 5, a - b = -7$$
$$\Rightarrow a = -\frac{16}{5}, b = \frac{19}{5}$$
$$\Rightarrow 5\mathbf{i} - 7\mathbf{j} = -\frac{16}{5}\mathbf{p} + \frac{19}{5}\mathbf{q}$$

3 $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$

So D has coordinates (0,5,-1).

4 **a** $\frac{8}{17}\mathbf{i} - \frac{15}{17}\mathbf{j}$

b $\frac{3}{\sqrt{59}}\mathbf{i} - \frac{1}{\sqrt{59}}\mathbf{j} - \frac{7}{\sqrt{59}}\mathbf{k}$

5 **a** $\pm\left(\frac{12}{\sqrt{61}}\mathbf{i} - \frac{15}{\sqrt{61}}\mathbf{j}\right)$

b $\pm\left(\frac{9}{13}\mathbf{i} - \frac{12}{13}\mathbf{j} - \frac{36}{13}\mathbf{k}\right)$

6 $AB = \sqrt{11}, AC = \sqrt{18}$ and $BC = \sqrt{29}$

$BC^2 = AB^2 + AC^2$

So by Pythagoras $\angle BAC = 90^\circ$.

9.3 Scalar product and its properties

1 Find the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

a $|\mathbf{a}| = \sqrt{2}, |\mathbf{b}| = 3, \theta = 45^\circ$

b $|\mathbf{a}| = 3, |\mathbf{b}| = 4, \theta = \dfrac{\pi}{5}$

2 If $\mathbf{a}=2\mathbf{i}-3\mathbf{b}+2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i}-5\mathbf{j}+\mathbf{k}$ find

a $\mathbf{a}.\mathbf{b}$

b the angle between **a** and **b**.

3 Given the vectors **a** and **b** such that **a** + **b** and **2a − b** are perpendicular and **a** − **b** and **4a** + **b** are perpendicular, find the angle between **a** and **b**.

Answers

1 **a** $\dfrac{3\sqrt{2}}{\sqrt{2}}=3.$

b $3\times4\times\cos\left(\dfrac{\pi}{5}\right)=9.708...\approx9.71$

2 **a** $\mathbf{a.b}=23$

b $\cos\theta=\dfrac{23}{\sqrt{17}\sqrt{35}}\Rightarrow\theta=19.45...\approx19.5^\circ$

3 $2|\mathbf{a}|^2-|\mathbf{b}|^2+\mathbf{a.b}=0\Rightarrow\mathbf{a.b}=|\mathbf{b}|^2-2|\mathbf{a}|^2$

$4|\mathbf{a}|^2-|\mathbf{b}|^2-3\mathbf{a.b}=0\Rightarrow3\mathbf{a.b}=4|\mathbf{a}|^2-|\mathbf{b}|^2$

So $4|\mathbf{a}|^2-|\mathbf{b}|^2=3|\mathbf{b}|^2-6|\mathbf{a}|^2$

$\Rightarrow|\mathbf{b}|=\dfrac{\sqrt{10}}{2}|\mathbf{a}|$

$\cos\theta=\dfrac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{\frac{1}{2}|\mathbf{a}|^2}{\frac{\sqrt{10}}{2}|\mathbf{a}|^2}=\dfrac{1}{\sqrt{10}}$

$\Rightarrow\theta=71.56...\approx71.6^\circ$

9.4 Vector equation of a line

1 Find a vector equation of the line passing through the points:

a (1,3) and (2,4)

b (3,-7) and (6,5)

2 Find a vector equation of the line that passes through the point (1,-4) and has normal vector $\mathbf{i}-3\mathbf{j}$.

3 The line $\mathrm{L}: \dfrac{x-2}{4}=\dfrac{y+3}{-5}$ and the point $\mathrm{T}: (4,2)$ are given.

Find a vector equation of the line that:

a is parallel to L and passes through T.

b passes through T and the intersection of L and the line $(8+\lambda)\mathbf{i}+(-2+3\lambda)\mathbf{j}$.

4 Find a vector equation of the line passing through the points (1,-2,3) and (4,1,-1).

5 Show that the lines

$\mathrm{L}_1: \dfrac{x-2}{3}=\dfrac{y-1}{4}=\dfrac{z+1}{2}$ and

$\mathrm{L}_2: \dfrac{x+1}{2}=\dfrac{y-2}{3}=z+5$

are skew.

6 Find the point of intersection of the lines

$\mathrm{L}_1: \dfrac{x-1}{2}=\dfrac{y+1}{3}=\dfrac{z-3}{4}$ and

$\mathrm{L}_2: \dfrac{x+1}{2}=y=\dfrac{z-1}{3}$.

Answers

1 **a** $\mathbf{r} = (\mathbf{i} + 3\mathbf{j}) + \lambda(\mathbf{i} + \mathbf{j})$

b $\mathbf{r} = (3\mathbf{i} - 7\mathbf{j}) + \lambda(3\mathbf{i} + 12\mathbf{j})$

2 $\mathbf{r} = (\mathbf{i} - 4\mathbf{j}) + \lambda(3\mathbf{i} + \mathbf{j})$

3 **a** $\mathbf{r} = (4\mathbf{i} + 2\mathbf{j}) + \lambda(4\mathbf{i} - 5\mathbf{j})$

b Lines intersect where

$$2 + 4\mu = 8 + \lambda \Rightarrow 4\mu - \lambda = 6$$
$$-3 - 5\mu = -2 + 3\lambda \Rightarrow 5\mu + 3\lambda = -1$$
$$\Rightarrow \mu = 1, \lambda = -2$$

so intersect at (6,-8)

and vector equation is $\mathrm{r}{=}(\mathbf{4i + 2j}) + \lambda(\mathbf{2i - 10j})$

4 $r = (\mathbf{i - 2j + 3k}) + \lambda(\mathbf{3i + 3j - 4k})$

5 Lines are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

From x and y equations

$$2{+}3\lambda{=}-1 + 2\mu \Rightarrow 3\lambda - 2\mu = -3$$
$$1{+}4\lambda{=}2 + 3\mu \Rightarrow 4\lambda - 3\mu = 1$$
$$\Rightarrow \lambda = -11, \mu = -15$$

Since these do not satisfy the z equation and the lines are clearly not parallel,

the lines must be skew.

6 Lines are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

From x and y equations

$$1+2\lambda = -1+2\mu \Rightarrow \lambda - \mu = -1$$
$$-1+3\lambda = \mu \Rightarrow 3\lambda - \mu = 1$$
$$\Rightarrow \lambda = 1, \mu = 2$$

Since these satisfy the z equation the lines meet at (3,2,7).

9.5 Vector product and properties

1 Find the vector product of:

a $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$

b $\frac{1}{2}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}$ and $\frac{2}{3}\mathbf{i}-\frac{1}{4}\mathbf{j}+\frac{1}{2}\mathbf{k}$.

2 Find the area of the parallelogram enclosed by the vectors

$3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}-4\mathbf{j}+5\mathbf{k}$.

3 Show that the points A (1,2,3), B (4,1,7), C (3,-2,0) and D (2,5,10) are coplanar using the mixed product.

4 $\overrightarrow{\mathbf{AB}}=2\mathbf{i}+3\mathbf{j}-\mathbf{k}, \overrightarrow{\mathbf{AC}}=3\mathbf{i}-\mathbf{j}+4\mathbf{k}$ and $\overrightarrow{\mathbf{AD}}=5\mathbf{i}+\mathbf{j}+7\mathbf{k}$.

Find the volume of the pyramid ABCD.

Answers

1 a

$$\begin{pmatrix}1\\-2\\3\end{pmatrix}\times\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-1\\10\\7\end{pmatrix}=\mathbf{-i}+\mathbf{10j}+\mathbf{7k}$$

b

$$\begin{pmatrix}\frac{1}{2}\\-\frac{1}{3}\\1\end{pmatrix}\times\begin{pmatrix}\frac{2}{3}\\-\frac{1}{4}\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{12}\\\frac{5}{12}\\\frac{7}{72}\end{pmatrix}=\frac{1}{12}\mathbf{i}+\frac{5}{12}\mathbf{j}+\frac{7}{72}\mathbf{k}$$

2

$$\begin{pmatrix}3\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\-4\\5\end{pmatrix}=\begin{pmatrix}3\\-13\\11\end{pmatrix}$$

So area $=\sqrt{299}$

3

$$\overrightarrow{AB}=\begin{pmatrix}3\\-1\\4\end{pmatrix},\overrightarrow{AC}=\begin{pmatrix}2\\-4\\-3\end{pmatrix},\overrightarrow{AD}=\begin{pmatrix}1\\3\\7\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}19\\17\\-10\end{pmatrix}$$

$$\begin{pmatrix}19\\17\\-10\end{pmatrix}.\begin{pmatrix}1\\3\\7\end{pmatrix}=0$$

So points are coplanar.

4

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 16 \\ 4 \\ -11 \end{pmatrix}$$

$$\begin{pmatrix} 16 \\ 4 \\ -11 \end{pmatrix} . \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} = 7$$

So volume = $\frac{7}{6}$.

9.6 Vector equation of a plane

1 Find the vector equation of the plane given by the vectors

$$\mathbf{u} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix} \text{ and the point } (7, 0, -3).$$

2 A plane passes through the points (2,1,3), (1,0,-1) and (5,4,7). Find:

a the vector equation of the plane

b the parametric equations of the plane

c the Cartesian equation of the plane.

3 Find the Cartesian equation of the plane in question 1.

4 Find the point of intersection of the planes $x + y + z = -4, 2x - y + 3z = -21$ and $3x + 4y + z = 4$.

Answers

1 $\mathbf{r}=\begin{pmatrix}7\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\2\end{pmatrix}+\mu\begin{pmatrix}-1\\3\\7\end{pmatrix}$

2 **a** $\mathbf{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\-4\end{pmatrix}+\mu\begin{pmatrix}3\\3\\4\end{pmatrix}$

b $x=2-\lambda+3\mu$

$y=1-\lambda+3\mu$

$z=3-4\lambda+4\mu$

c $x-y=1$

3 $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}1\\-23\\10\end{pmatrix}$

So Cartesian equation is $x-23y+10z=-23$

4 From first two equations $3x+4z=-25$

From last two equations $11x+13z=-80$

So $-5x=-5\Rightarrow x=1, z=-7, y=2$

So point of intersection is (1,2,-7)

9.7 Lines, planes and angles

1 Find the point of intersection of the line

$x-2=\frac{3-y}{4}=\frac{z+1}{6}$ and the plane $2x+y+4z=25$.

2 Find the values of the real parameter *m* such that the line

$\frac{x}{m}=\frac{y-1}{3}=\frac{z+2}{-2}$ is parallel to the plane $mx+my+5z=7$.

3 Find the distance between the point (1,2,-3) and the plane

$x-3y+4z=-32$.

4 **a** Show that the planes $x-y+z=4$ and $2x-2y+2z=6$ are parallel.

b Find the distances between the two planes.

5 Find the distance between the parallel lines

$\mathbf{r}=(1+2\lambda)\mathbf{i}+(3-\lambda)\mathbf{j}+(4\lambda-1)\mathbf{k}$ and

$\frac{x+3}{2}=5-y=\frac{2-z}{4}$.

6 Show that the lines

$\mathbf{r}=\begin{pmatrix}4\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $x-2=\frac{y-3}{2}=\frac{z-7}{4}$ are skew.

Find the distance between them.

7 **a** Find the equation of the line of intersection of the planes

$x+y-2z=4$ and $x+y-2z=4$.

b Determine the angle between the two planes.

8 Show that the lines

$\mathbf{r}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}0\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-3\end{pmatrix}$ intersect and

find the angle between the two lines.

9 Find the angle between the line

$\frac{x+1}{3}=\frac{1-y}{2}=\frac{z}{6}$ and the plane $x+2y-4z=7$.

Answers

1 $\mathbf{r} = (2+\lambda)\mathbf{i} + (3-4\lambda)\mathbf{j} + (-1+6\lambda)\mathbf{k}$

So intersects plane where

$$4+2\lambda+3-4\lambda-4+24\lambda=25$$
$$\Rightarrow 22\lambda+3=25$$
$$\Rightarrow \lambda=1$$

So intersect at (3,-1,5)

2 Parallel to plane if

$$\begin{pmatrix} m \\ 3 \\ -2 \end{pmatrix} . \begin{pmatrix} m \\ m \\ 5 \end{pmatrix} = 0$$
$$\Rightarrow m^2+3m-10=0$$
$$\Rightarrow m=2 \text{ or } -5.$$

3 Consider the line

$$\text{r}=\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

So meets plane where

$$2+4\lambda-6+9\lambda-12+16\lambda=-32$$

$$\Rightarrow \lambda=-\frac{16}{29}$$

And meets plane at $\left(-\frac{3}{29}, \frac{106}{29}, -\frac{151}{29}\right)$

So distance $=\sqrt{\left(1+\frac{3}{29}\right)^2+\left(2-\frac{106}{29}\right)^2+\left(-3+\frac{151}{29}\right)^2}$

$$=3.029\ldots \approx 3.03$$

4 **a** Planes are clearly parallel by looking at the normal vectors.

b Consider the point (2,0,2) on $x-y+z=4$

Normal through that point is

$$r=\begin{pmatrix}2\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

So meets the second plane where

$$4+2\lambda+2\lambda+4+2\lambda=6\Rightarrow\lambda=-\frac{1}{3}$$

$$\Rightarrow \text{distance} = \frac{\sqrt{3}}{3}$$

5 Consider vector joining (1,3,-1) to a point on the second line.

$$\begin{pmatrix}1+3-2\lambda\\3-5+\lambda\\-1-2-4\lambda\end{pmatrix}=\begin{pmatrix}4-2\lambda\\-2+\lambda\\-3-4\lambda\end{pmatrix}$$

This is normal to the line when

$$\begin{pmatrix}4-2\lambda\\-2+\lambda\\-3-4\lambda\end{pmatrix}.\begin{pmatrix}2\\1\\-4\end{pmatrix}=0$$

$$\Rightarrow 8-4\lambda+2-\lambda-12-16\lambda=0$$

$$\Rightarrow\lambda=-\frac{2}{21}$$

So distance

$$=\sqrt{\left(4+\frac{4}{21}\right)^2+\left(-2-\frac{2}{21}\right)^2+\left(-3+\frac{8}{21}\right)^2}$$

$$=5.367...\approx 5.37$$

6 Lines are:

$$\mathbf{r}=\begin{pmatrix}2\\3\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\4\end{pmatrix},\mathbf{r}=\begin{pmatrix}4\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\1\\1\end{pmatrix}$$

If lines intersect then

$$\lambda+\mu=2$$
$$2\lambda-\mu=-1$$
$$4\lambda-\mu=-6$$

First two equations $\Rightarrow \lambda=\frac{1}{3}, \mu=\frac{5}{3}$

But these do not satisfy the third equation so, since the lines are clearly not parallel,

they must be skew.

Vector joining 2 points is given by

$$\begin{pmatrix}\lambda+\mu-2\\2\lambda-\mu+1\\4\lambda-\mu+6\end{pmatrix}$$

If perpendicular to both lines then

$$\begin{pmatrix}\lambda+\mu-2\\2\lambda-\mu+1\\4\lambda-\mu+6\end{pmatrix}.\begin{pmatrix}1\\2\\4\end{pmatrix}=0\Rightarrow 21\lambda-7\mu+24=0$$

$$\begin{pmatrix}\lambda+\mu-2\\2\lambda-\mu+1\\4\lambda-\mu+6\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}=0\Rightarrow 7\lambda-3\mu+5=0$$

$$\Rightarrow \lambda=-\frac{37}{14}, \mu=-\frac{9}{2}$$

So distance

$$=\sqrt{\left(-\frac{37}{14}+\frac{9}{2}-2\right)^2+\left(2\times-\frac{37}{14}+\frac{9}{2}+1\right)^2+\left(4\times-\frac{37}{14}+\frac{9}{2}+6\right)^2}$$
$$=0.2672\ldots\approx 0.267$$

7 a (0,0,-2) and (1,1,-1) lie on line of intersection

So $\mathbf{r}=\begin{pmatrix}0\\0\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$

b $\cos\theta=-\dfrac{3}{\sqrt{6}\sqrt{14}}$

So taking acute angle $\theta=70.89\ldots\approx 70.9^{\circ}$.

8 Lines clearly intersect when $\lambda=-1, \mu=0$

Angle is given by $\cos\theta=\dfrac{2}{\sqrt{2}\times\sqrt{11}}$

$\Rightarrow\theta=64.76\approx 64.8^{\circ}$.

9 Angle between line and normal is given by

$\cos\theta=\dfrac{-17}{7\sqrt{21}}\Rightarrow\theta=122.00\ldots$

So $=58.00$ taking acute angle

So angle between line and plane $\approx 32.0^{\circ}$.

9.8 Application of vectors

1 A particle moves with constant speed in such a way that, at time t seconds, its position in metres is given by

$$r = \begin{pmatrix} 7+4t \\ 2-3t \\ -5+t \end{pmatrix}.$$

a Find its position at $t = 2$.

b Find the speed of the particle.

c When will it cross the plane $x + y + z = 5$?

2 Two speedboats are sailing in such a way that their positions are given by

$$\mathbf{r}_1 = \begin{pmatrix} 200+4.2t \\ 100-4.8t \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} 146+13.8t \\ 61-5.7t \end{pmatrix}$$

Show that they do not collide.

Answers

1 **a**

$$r = \begin{pmatrix} 15 \\ -4 \\ -3 \end{pmatrix}$$

b Speed $= \sqrt{4^2 + (-3)^2 + 1^2}$

$$= 5.099... \approx 5.10\text{ms}^{-1}$$

c $7 + 4t + 2 - 3t + -5 + t = 5$

$$\Rightarrow 4 + 2t = 5$$
$$\Rightarrow t = 0.5s$$

2 $200 + 4.2t_1 = 146 + 13.8t_2 \Rightarrow 4.2t_1 - 13.8t_2 = -54$

$$100 - 4.8t_1 = 61 - 5.7t_2 \Rightarrow 4.8t_1 - 5.7t_2 = 39$$

$$\Rightarrow t_1 = 20, t_2 = 10$$

So, since times are different, they do not collide.

10.1 Forms of a complex number

1 Given the following module and argument, draw the corresponding complex number on a Cartesian plane.

a $|z| = 1$, $\arg(z) = \frac{\pi}{2}$

b $|z| = 2$, $\arg(z) = \frac{3\pi}{4}$

c $|z| = 2$, $\arg(z) = \frac{11\pi}{6}$

2 Convert the following complex numbers to polar form.

a $z = 6\text{i}$

b $z = 6\sqrt{3} - 6\text{i}$

c $z = -3 - 4\text{i}$

3 Convert the following complex numbers to Cartesian form.

a $z = 4e^{\frac{-\pi\text{i}}{3}}$

b $z = 4\text{cis}240°$

c $z = 8\text{cis}\frac{\pi}{4}$

d $z = 5\text{cis}(-30°)$

4 Given $z = -1 + \sqrt{3}\text{i}$, find z^* and $-z^*$ in polar form.

5 If $u = 2 + 4\text{i}$ and $v = 4 + m\text{i}$, find m when $|v| = 4|u|$.

6 Simplify $\text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(\frac{7\pi}{10}\right)$.

Answers

1 **a** This is a circle centered at the origin with radius of 1 and ray drawn from the origin at an angle of $\frac{\pi}{2}$.

b This is a circle centered at the origin with radius of 2 and ray drawn from the origin at an angle of $\frac{3\pi}{4}$.

c This is a circle centered at the origin with radius of 2 and ray drawn from the origin at an angle of $\frac{11\pi}{6}$.

2 **a** Modulus → $|0+6i| = \sqrt{(6)^2} = 6$

Argument → $\arctan\left(\frac{6}{0}\right) = \arctan(\textit{undefined}) = \frac{\pi}{2}$

$6i$ → $6\text{cis}\frac{\pi}{2}$

b Modulus → $|6\sqrt{3}-6i| = \sqrt{\left(6\sqrt{3}\right)^2 + (-6)^2} = \sqrt{108+36} = \sqrt{144} = 12$

Argument → $\arctan\left(\frac{-6}{6\sqrt{3}}\right) = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$

$6\sqrt{3}-6i$ → $12\text{cis}\frac{5\pi}{6}$

c Modulus → $|-3-4i| = \sqrt{(-3)^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Argument → $\arctan\left(\frac{-4}{-3}\right) \approx 0.927$

$-3-4i$ → $5\textit{cis}0.927$

3 **a** $z = 4e^{\frac{-\pi i}{3}} = 4\text{cis}\left(\frac{-\pi}{3}\right) = 5\cos\left(\frac{5\pi}{3}\right) + 5\text{i}\sin\left(\frac{5\pi}{3}\right) = 5\left(\frac{1}{2}\right) + 5\text{i}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5}{2} - \frac{5\sqrt{3}}{2}\text{i}$

b $z = 4\text{cis}240° = 4\cos\left(\frac{4\pi}{3}\right) + 4\text{i}\sin\left(\frac{4\pi}{3}\right) = 4\left(\frac{-1}{2}\right) + 4\text{i}\left(\frac{-\sqrt{3}}{2}\right) = -2-2\sqrt{3}\text{i}$

c $z = 8\text{cis}\frac{\pi}{4} = 8\cos\left(\frac{\pi}{4}\right) + 8\text{i}\sin\left(\frac{\pi}{4}\right) = 8\left(\frac{\sqrt{2}}{2}\right) + 8\text{i}\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2} + 4\sqrt{2}\text{i}$

d $z = 5\text{cis}(-30°) = 5\cos\left(\frac{11\pi}{6}\right) + 5\text{i}\sin\left(\frac{11\pi}{6}\right) = 5\left(\frac{\sqrt{3}}{2}\right) + 5\text{i}\left(-\frac{1}{2}\right) = \frac{5\sqrt{3}}{2} - \frac{5}{2}\text{i}$

4 $z^* = -1-\sqrt{3}i$

$|-1-\sqrt{3}i| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\arg\left(-1-\sqrt{3}i\right)=arctan\left(\frac{-\sqrt{3}}{-1}\right)=arctan\left(\sqrt{3}\right)=\frac{4\pi}{3}$$

$$z^{*}=2cis\frac{4\pi}{3}$$

$$-z^{*}=1+\sqrt{3}i$$

$$\left|1+\sqrt{3}i\right|=\sqrt{(1)^{2}+\left(\sqrt{3}\right)^{2}}=\sqrt{1+3}=\sqrt{4}=2$$

$$\arg\left(1+\sqrt{3}i\right) = arctan\left(\sqrt{3}\right)=\frac{\pi}{3}$$

$$-z^{*}=2cis\frac{\pi}{3}$$

5 If $u=2+4i$ and $v=4+mi$, find m when $|v|=4|u|$.

$$|u|=\sqrt{4+16}=2\sqrt{5}$$

$$|v|=\sqrt{16+m^{2}}$$

$$\sqrt{16+m^{2}}=4\left(2\sqrt{5}\right)=8\sqrt{5}$$

$$\left(\sqrt{16+m^{2}}\right)^{2}=\left(8\sqrt{5}\right)^{2}$$

$$16+m^{2}=64(5)=320$$

$$m^{2}=304$$

$$m=\pm\sqrt{304}$$

6 $cis\left(\frac{2\pi}{5}\right)cis\left(\frac{7\pi}{10}\right)=cis\left(\frac{2\pi}{5}+\frac{7\pi}{10}\right)=cis\left(\frac{15\pi}{10}\right)=cis\left(\frac{3\pi}{2}\right)=cos\left(\frac{3\pi}{2}\right)+isin\left(\frac{3\pi}{2}\right)=-i$

10.2 Operations with complex numbers in polar form

1 Simplify $(3-4i)(2+3i)+(2-2i)^2$.

2 If $z_1=-2\text{cis}\frac{\pi}{4}$ and $z_2=3\text{cis}\frac{5\pi}{6}$ find z_1z_2.

3 Given $z_1=5\text{cis}\frac{\pi}{6}=$ and $z_2=-2\text{cis}\frac{7\pi}{4}$, find

a z_1z_2 in polar form

b z_1z_2 in Euler's form

c $\frac{z_1}{z_2}$

d $\frac{z_2}{z_1}$

4 If $u=1+3i$ and $v=2-i$, find $\frac{u}{v}$ in polar form.

5 For question 4, find $\frac{u^*}{v}$ in polar form.

6 Simplify the following in Cartesian form.

a $(3+i)(2-i)^2-i$

b $\frac{i^3}{2+i}$

c $\frac{4-3i^2+2i}{(2-2i^3)^2}$

7 Given $|z|+z=3-i$, find z.

8 Consider $z_1=cis(225°)$ and $z_2=\frac{\sqrt{3}}{2}-\frac{1}{2}i$.

a Find $\frac{z_1}{z_2}$ in both polar form and Cartesian form.

b Hence, find the exact value for $\sin 225°$.

9 Let u,v be two non-zero complex numbers. If $uv^* + u^*v = 0$, show that $\frac{u}{v}$ has no real part.

10 Find the sum of $1 - i + i^2 - i^3 + \ldots + i^{100} - i^{101}$.

Answers

1 $(3-4i)(2+3i)+(2-2i)^2$

$(3-4i)(2+3i)+(2-2i)(2-2i)$

$(6+9i-8i+12)+(4-4i-4i-4)$

$18+i-8i$

$18-7i$

2 $z_1z_2=-6\text{cis}\left(\frac{\pi}{4}+\frac{5\pi}{6}\right)$

$z_1z_2=-6\text{cis}\left(\frac{3\pi}{12}+\frac{10\pi}{12}\right)$

$z_1z_2=-6\text{cis}\left(\frac{13\pi}{12}\right)$

3 **a** $z_1z_2=-10\text{cis}\left(\frac{\pi}{6}+\frac{7\pi}{4}\right)=-10\text{cis}\left(\frac{2\pi}{12}+\frac{21\pi}{12}\right)=-10\text{cis}\left(\frac{23\pi}{12}\right)$

b $z_1z_2=-10e^{\left(\frac{23\pi}{12}\text{i}\right)}$

c $\frac{z_1}{z_2}=\frac{5\text{cis}\frac{\pi}{6}}{-2\text{cis}\frac{7\pi}{4}}=-\frac{5}{2}\text{cis}\left(\frac{\pi}{6}-\frac{7\pi}{4}\right)=-\frac{5}{2}\text{cis}\left(\frac{2\pi}{12}-\frac{21\pi}{12}\right)=-\frac{5}{2}\text{cis}\left(-\frac{19\pi}{12}\right)=-\frac{5}{2}\text{cis}\left(\frac{5\pi}{12}\right)$

d $\frac{z_2}{z_1}=\frac{-2\text{cis}\frac{7\pi}{4}}{5\text{cis}\frac{\pi}{6}}=-\frac{2}{5}\text{cis}\left(\frac{7\pi}{4}-\frac{\pi}{6}\right)=-\frac{2}{5}\text{cis}\left(\frac{21\pi}{12}-\frac{2\pi}{12}\right)=-\frac{2}{5}\text{cis}\left(\frac{19\pi}{12}\right)$

4 $|u|=\sqrt{1^2+3^2}=\sqrt{10}$

$\arg(u)=\arctan(3)=1.24905$

$u=\sqrt{10}\text{cis}(1.24905\ldots)$

$|v|=\sqrt{2^2+(-1)^2}=\sqrt{5}$

$\arg(v)=\arctan\left(\frac{-1}{2}\right)=5.81954\ldots$

$v=\sqrt{5}\text{cis}(5.81954\ldots)$

$\frac{u}{v}=\frac{\sqrt{10}}{\sqrt{5}}\text{cis}(1.24905\ldots-5.81954\ldots)$

$\frac{u}{v} = \sqrt{2}\text{cis}(-4.57049\ldots)$

$\frac{u}{v} \approx \sqrt{2}\text{cis}(1.71)$

5 $u^* = 1 - 3i$

$|u^*| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\arg(u^*) = \arctan(-3) = 5.03414\ldots$

$u^* = \sqrt{10}\text{cis}(5.03414\ldots)$

$\frac{u^*}{v} = \frac{\sqrt{10}}{\sqrt{5}}\text{cis}(5.03414\ldots - 5.81954\ldots)$

$\frac{u^*}{v} = \sqrt{2}\text{cis}(-0.7854\ldots)$

$\frac{u^*}{v} \approx \sqrt{2}\text{cis}(5.50)$

6 **a** $(3+i)(2-i)^2 - i$

$(3+i)(4-2i-2i-1) - i$

$(3+i)(3-4i) - i$

$(9-12i+3i+4) - i$

$13 - 9i - i$

$13 - 10i$

b $\frac{i^3}{2+i} = \frac{-i}{(2+i)(2-i)} = \frac{-i(1)}{(2+i)(2-i)} = \frac{-i(2-1)}{(2+i)(2-i)} = \frac{-1-2i}{5} = -\frac{1}{5} - \frac{2}{5}i$

c $\frac{4-3i^2+2i}{(2-2i^3)^2} = \frac{4+3+2i}{(2+2i)^2} = \frac{7+2i}{4+4i+4i-4} = \frac{7+2i}{8i} = \frac{(7+2i)(-i)}{(8i)(-i)} = \frac{2-7i}{8} = \frac{1}{4} - \frac{7}{8}i$

7 Let $z = a + bi$

$|a+bi| + a + bi = 3 - i$

$\left(\sqrt{a^2+b^2} + a\right) + bi = 3 - i$

Equating imaginary parts:

$bi = -i$

$b = -1$

Equating real parts:

$\sqrt{a^2 + b^2} + a = 3$

$\sqrt{a^2 + (-1)^2} + a = 3$

$\sqrt{a^2 + 1} = 3 - a$

$\left(\sqrt{a^2 + 1}\right)^2 = (3 - a)^2$

$a^2 + 1 = 9 - 6a + a^2$

$1 = 9 - 6a$

$-8 = -6a$

$a = \frac{4}{3}$

$z = \frac{4}{3} - i$

8 **a** $z_1 = e^{225^o} = \text{cis}225^o = \cos 225^o + i\sin 225^o = \frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$|z_2| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\arg(z_2) = \arctan\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \arctan\left(\frac{-1}{\sqrt{3}}\right) = 330^o$

$z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i = \text{cis}330^o = e^{330^o}$

$\frac{z_1}{z_2} = \frac{e^{225^o}}{e^{330^o}} = e^{-105^\circ} = e^{225^\circ} = cos225^\circ + isin225^\circ$

$\frac{z_1}{z_2} = \frac{\frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2} - \frac{1}{2}i} \cdot \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}i}{\frac{\sqrt{3}}{2} + \frac{1}{2}i}$

$$\frac{z_1}{z_2} = \frac{\frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4}i - \frac{\sqrt{6}}{4}i + \frac{\sqrt{2}}{4}}{\frac{3}{4} + \frac{1}{4}} = \frac{-\sqrt{6} + \sqrt{2}}{4} - \frac{\sqrt{6} + \sqrt{2}}{4}i$$

b $\sin 225° = -\frac{\sqrt{6} + \sqrt{2}}{4}$

9 Let $u = a + bi$ and $v = c + di$.

Therefore $u^* = a - bi$ and $v^* = c - di$.

$uv^* + u^*v = 0$

$(a + bi)(c - di) + (a - bi)(c + di) = 0$

$ac + adi + bci + bd + ac - dai - bci + bd = 0$

$2ac + 2bd = 0$

$ac + bd = 0$

$$\frac{u}{v} = \frac{a + bi}{c + di}$$

$$\frac{u}{v} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}$$

$$\frac{u}{v} = \frac{ac - adi + bci + bd}{c^2 + d^2}$$

$$\frac{u}{v} = \frac{ac + bd}{c^2 + d^2} + \frac{bci - adi}{c^2 + d^2}i$$

$$Re\left(\frac{u}{v}\right) = \frac{ac + bd}{c^2 + d^2}$$

But since $ac + bd = 0$, $Re\left(\frac{u}{v}\right) = 0$.

10 It is a geometric sequence since $r = \frac{i^2}{-i} = \frac{-i^3}{i^2} = -i$

$$S_{102} = \frac{1 - (-i)^{102}}{1 - -i} = \frac{1 - (-1)}{1 + i} = \frac{2}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{2(1 - i)}{2} = 1 - i$$

10.3 Powers and roots of complex numbers in polar form

1 Use De Moivre's theorem to calculate $\left(\sqrt{3}+i\right)^6$.

2 Simplify $\left(\sqrt{2}-i\sqrt{2}\right)^5$.

3 Find the two square roots of $u=-5+12i$.

4 Show that if $z=\text{cis}\theta$, then $\dfrac{z^2-1}{z^2+1}=i\tan\theta$.

5 Given that $z_1=1-i$ and $z_2=3e^{\frac{5\pi i}{6}}$, find $z_1^2\cdot\left(z_2^*\right)^3$.

6 Solve $z^3=-64i$.

7 Solve $z^4=-4-4i$.

8 Given then $z=cos\theta+isin\theta$, show that:

a $\text{Im}\left(z^n+\dfrac{1}{z^n}\right)=0, z\neq-1$

b $\text{Re}\left(\dfrac{z-1}{z+1}\right)=2\cos(n\theta), n\in\mathbb{Z}^+$

9 Determine the solution of the equation $(z+2i)^3=216i$ giving your answer in Cartesian form.

10 If $z=3-3i$ and $u=\text{cis}\left(\dfrac{7\pi}{6}\right)$, find z^3u^4 in Euler's form.

11 Find all values of n such that $z=(2+2i)^n$ is a real number.

12 Given $u=\left(-1-\sqrt{3}i\right)^a$ and $z=(1+i)^b$, find the smallest positive integers a and b so that $u=z$.

13 a If $v=1-\cos2x+i\sin2x$, find the modulus and argument of v in terms of x.

b Hence, find the cube roots of v in Cartesian form.

Answers

1 Let $u = \sqrt{3} + i$

$|u| = \sqrt{3+1} = 2$

$\arg(u) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$u = 2\text{cis}\left(\frac{\pi}{6}\right)$

$u^6 = 2^6\left(\cos 6\left(\frac{\pi}{6}\right) + i\sin 6\left(\frac{\pi}{6}\right)\right)$

$u^6 = 64\cos\pi + 64i\sin\pi$

$u^6 = 64(-1) + 64i(0)$

$u^6 = -64$

$\left(\sqrt{3} + i\right)^6 = -64$

2 Let $u = \sqrt{2} - i\sqrt{2}$

$|u| = \sqrt{2+2} = 2$

$\arg(u) = \arctan\left(\frac{-\sqrt{2}}{\sqrt{2}}\right) = \arctan(-1) = \frac{7\pi}{4}$

$u = 2\text{cis}\left(\frac{7\pi}{4}\right)$

$u^5 = 2^5\left(\cos 5\left(\frac{7\pi}{4}\right) + i\sin 5\left(\frac{7\pi}{4}\right)\right)$

$u^5 = 32\left(\cos\left(\frac{35\pi}{4}\right) + i\sin\left(\frac{35\pi}{4}\right)\right)$

$u^5 = 32\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$u^5 = 32\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$u^5 = -16\sqrt{2} + 16\sqrt{2}i$

3 $u = -5 + 12i$

$|u| = \sqrt{25+144} = 13$

$\arg(u) = \arctan\left(\frac{12}{-5}\right) \approx 1.97$

$u \approx 13\text{cis}(1.97)$

$\sqrt{u} \approx \sqrt{13}\text{cis}\left(\frac{1}{2}\cdot 1.97\right)$

$\sqrt{u} \approx \sqrt{13}\text{cis}(0.985)$

$\sqrt{u} \approx \sqrt{13}\cos(0.985) + \sqrt{13}i\sin(0.0985)$

$\sqrt{u} \approx 2 + 3i$

To find n roots, they will be $\frac{2\pi}{n}$ apart, so for 2 roots, they will be π apart.

Therefore the second root $\approx \sqrt{13}\text{cis}(0.985 + \pi)$ or $-2 - 3i$.

The two square roots are $2 + 3i$ and $-2 - 3i$.

4 $z = \text{cis}\theta = \cos\theta + i\sin\theta$

$z^2 = \text{cis}2\theta = \cos 2\theta + i\sin 2\theta$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{\cos 2\theta + i\sin 2\theta - 1}{\cos 2\theta + i\sin 2\theta + 1}$$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{i\sin 2\theta - (1 - \cos 2\theta)}{i\sin 2\theta + (1 + \cos 2\theta)}$$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{i(2\sin\theta\cos\theta) - (2\sin^2\theta)}{i(2\sin\theta\cos\theta) + (2 - 2\sin^2\theta)}$$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{i(2\sin\theta\cos\theta) + i^2(2\sin^2\theta)}{i(2\sin\theta\cos\theta) + 2(1 - \sin^2\theta)}$$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{2i\sin\theta(\cos\theta + i\sin\theta)}{i(2\sin\theta\cos\theta) + 2\cos^2\theta}$$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{2i\sin\theta(\cos\theta + i\sin\theta)}{2\cos\theta(\cos\theta + i\sin\theta)}$$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{i\sin\theta}{\cos\theta}$$

$$\frac{z^2 - 1}{z^2 + 1} = i\tan\theta$$

5 $z_1 = 1 - i$

$|z_1| = \sqrt{1+1} = \sqrt{2}$

$\arg(z_1) = \arctan(-1) = \frac{7\pi}{4}$

$$z_1 = \sqrt{2}\cos\left(\frac{7\pi}{4}\right) + i\sqrt{2}\sin\left(\frac{7\pi}{4}\right)$$

$$z_1^2 = 2\cos\left(\frac{14\pi}{4}\right) + 2i\sin\left(\frac{14\pi}{4}\right)$$

$$z_1^2 = 2\cos\left(\frac{3\pi}{2}\right) + i2\sin\left(\frac{3\pi}{2}\right)$$

$$z_1^2 = -2i$$

$$z_2 = 3\cos\left(\frac{5\pi}{6}\right) + 3i\sin\left(\frac{5\pi}{6}\right)$$

$$z_2^* = 3\cos\left(\frac{5\pi}{6}\right) - 3i\sin\left(\frac{5\pi}{6}\right)$$

$$\left(z_2^*\right)^3 = 27\cos\left(\frac{15\pi}{6}\right) - i27\sin\left(\frac{15\pi}{6}\right)$$

$$\left(z_2^*\right)^3 = 27\cos\left(\frac{\pi}{2}\right) - i27\sin\left(\frac{\pi}{2}\right)$$

$$\left(z_2^*\right)^3 = -27i$$

$$z_1^2 \cdot \left(z_2^*\right)^3 = (-2i)(-27i) = -54$$

6 From GDC: $x = 4i, -2\sqrt{3} - 2i, 2\sqrt{3} - 2i$

7 $z^4 = -4 - 4i$

$$z = (-4 - 4i)^{\frac{1}{4}}$$

Let $u = -4 - 4i$.

$$|u| = \sqrt{16 + 16} = \sqrt{32} = 2^{\frac{5}{2}}$$

$$\arg(u) = \arctan\left(\frac{-4}{-4}\right) = \arctan(1) = \frac{5\pi}{4}$$

$$u = 2^{\frac{5}{2}}\operatorname{cis}\left(\frac{5\pi}{4}\right)$$

$$z = \left(2^{\frac{5}{2}}\operatorname{cis}\left(\frac{5\pi}{4}\right)\right)^{\frac{1}{4}}$$

$$z = \left(2^{\frac{5}{2}}\right)^{\frac{1}{4}}\left(\cos\left(\frac{5\pi}{16}\right) + i\sin\left(\frac{5\pi}{16}\right)\right)$$

$$z = 2^{\frac{5}{8}}\left(\cos\left(\frac{5\pi}{16}\right) + i\sin\left(\frac{5\pi}{16}\right)\right)$$

To find n roots, they will be $\frac{2\pi}{n}$ apart, so for 4 roots, they will be $\frac{\pi}{2}$ apart

$$z = 2^{\frac{5}{8}}\left(\cos\left(\frac{5\pi}{16}+\frac{\pi}{2}\right)+i\sin\left(\frac{5\pi}{16}+\frac{\pi}{2}\right)\right) = 2^{\frac{5}{8}}\left(\cos\left(\frac{5\pi}{16}+\frac{8\pi}{16}\right)+i\sin\left(\frac{5\pi}{16}+\frac{8\pi}{16}\right)\right) = 2^{\frac{5}{8}}\left(\cos\left(\frac{13\pi}{16}\right)+i\sin\left(\frac{13\pi}{16}\right)\right)$$

Likewise,

$$z = 2^{\frac{5}{8}}\left(\cos\left(\frac{5\pi}{16}\right)+i\sin\left(\frac{5\pi}{16}\right)\right), 2^{\frac{5}{8}}\left(\cos\left(\frac{13\pi}{16}\right)+i\sin\left(\frac{13\pi}{16}\right)\right), 2^{\frac{5}{8}}\left(\cos\left(\frac{21\pi}{16}\right)+i\sin\left(\frac{21\pi}{16}\right)\right), 2^{\frac{5}{8}}\left(\cos\left(\frac{29\pi}{16}\right)+i\sin\left(\frac{29\pi}{16}\right)\right)$$

8 **a** $z = \cos\theta + i\sin\theta$

$$z^n = \cos(n\theta) + i\sin(n\theta)$$

$$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$

$$z^n + \frac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$$

$$z^n + \frac{1}{z^n} = \cos(n\theta) + \cos(-n\theta) + i\left(\sin(n\theta) + \sin(-n\theta)\right)$$

$$Im\left(z^n + \frac{1}{z^n}\right) = \sin(n\theta) + \sin(-n\theta) = \sin(n\theta) - \sin(n\theta) = 0$$

b $Re\left(z^n + \frac{1}{z^n}\right)\ \cos(n\theta) + \cos(-n\theta) = \cos(n\theta) + \cos(n\theta) = 2\cos(n\theta)$

9 $(z+2i)^3 = 216i$

$$(z+2i)^3 = 216\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$$

$$z+2i = \left(216\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)\right)^{\frac{1}{3}}$$

$$z+2i = 6\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right) = 6\left(\left(\frac{\sqrt{3}}{2}\right)+i\left(\frac{1}{2}\right)\right) = 3\sqrt{3}+3i$$

$$z = 3\sqrt{3}+i$$

To find n roots, they will be $\frac{2\pi}{n}$ apart, so for 3 roots, they will be $\frac{2\pi}{3}$ apart.

Second root:

$$z+2i = 6\left(\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\right)$$

$$z+2i = 6\left(\cos\left(\frac{\pi}{6}+\frac{4\pi}{6}\right)+i\sin\left(\frac{\pi}{6}+\frac{4\pi}{6}\right)\right)$$

$z+2i=6\left(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right)$

$z+2i=6\left(\left(\frac{-\sqrt{3}}{2}\right)+i\left(\frac{1}{2}\right)\right)=-3\sqrt{3}+3i$

Third root:

$z+2i=6\left(\cos\left(\frac{5\pi}{6}+\frac{2\pi}{3}\right)+i\sin\left(\frac{5\pi}{6}+\frac{2\pi}{3}\right)\right)$

$z+2i=6\left(\cos\left(\frac{5\pi}{6}+\frac{4\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}+\frac{4\pi}{6}\right)\right)$

$z+2i=6\left(\cos\left(\frac{9\pi}{6}\right)+i\sin\left(\frac{9\pi}{6}\right)\right)$

$z+2i=6\left(\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right)$

$z+2i=-6i$

$z=-8i$

$\therefore z=-3\sqrt{3}+3i, 3\sqrt{3}+i, -8i$

10 $z=3-3i$

$|z|=\sqrt{3^2+(-3)^2}=\sqrt{9+9}=3\sqrt{2}$

$\arg(z)=\arctan(-1)=\frac{7\pi}{4}$

$z^3=\left(3\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{4}\right)\right)^3$

$z^3=\left(3\sqrt{2}\right)^3\operatorname{cis}\left(\frac{21\pi}{4}\right)$

$z^3=27\sqrt{2^3}\operatorname{cis}\left(\frac{5\pi}{4}\right)$

$z^3=27\sqrt{2^3}e^{\frac{5\pi}{4}i}$

$u=\operatorname{cis}\left(\frac{7\pi}{6}\right)$

$u^4=\operatorname{cis}\left(\frac{28\pi}{6}\right)$

$u^4=\operatorname{cis}\left(\frac{2\pi}{3}\right)$

$u^4=e^{\frac{2\pi}{3}i}$

$$z^3u^4 = \left(27\sqrt{2^3}e^{\frac{5\pi}{4}i}\right)\left(e^{\frac{2\pi}{3}i}\right)$$

$$z^3u^4 = 27\sqrt{2^3}e^{\frac{23\pi}{12}i}$$

11 Let $u = 2 = 2i$

$$|u| = \sqrt{2^2 + 2^2} = \sqrt{4+4} = 2\sqrt{2}$$

$$\arg(u) = \arctan(1) = \frac{\pi}{4}$$

$$z = (2+2i)^n$$

$$z = \left(2\sqrt{2}\right)^n cos\frac{n\pi}{4} + \left(2\sqrt{2}\right)^n isin\frac{n\pi}{4}$$

For z to be a real number, $sin\frac{n\pi}{4} = 0$ for $n = 0, \pm4, \pm8, \ldots$

$$\frac{n\pi}{4} = k\pi, k\epsilon\mathbb{Z}$$

$$n = 4k, k\epsilon\mathbb{Z}$$

12 $u = \left(-1-\sqrt{3}i\right)^a$

$$\left|-1-\sqrt{3}i\right| = \sqrt{1+3} = 2$$

$$\arg\left(-1-\sqrt{3}i\right) = \arctan\left(\sqrt{3}\right) = \frac{4\pi}{3}$$

$$u = \left(2\text{cis}\frac{4\pi}{3}\right)^a$$

$$u = 2^a\text{cis}\frac{4a\pi}{3}$$

$$z = (1+i)^b$$

$$|1+i| = \sqrt{1+1} = \sqrt{2}$$

$$\arg(1+i) = \arctan(1) = \frac{\pi}{4}$$

$$z = \left(\sqrt{2}\text{cis}\frac{\pi}{4}\right)^b$$

$$z = \sqrt{2}^b\text{cis}\frac{b\pi}{4}$$

$$u = z$$

$$2^a\text{cis}\frac{4a\pi}{3} = \sqrt{2}^b\text{cis}\frac{b\pi}{4}$$

Equating real parts:

$$2^a = \sqrt{2}^b$$

$$2^a = 2^{\frac{1}{2}b}$$

$$a = \frac{1}{2}b$$

$$2a = b$$

Equating imaginary parts:

$$\frac{4a\pi}{3} = \frac{b\pi}{4} + 2k\pi, k \in \mathbb{Z}$$

$$\frac{4a\pi}{3} - \frac{2a\pi}{4} = 2k\pi$$

$$\frac{16a\pi}{12} - \frac{6a\pi}{12} = 2k\pi$$

$$\frac{10a\pi}{12} = \frac{24k\pi}{12}$$

$$10a = 24k$$

$$a = \frac{12k}{5}$$

The smallest value of k such that a is an integer is 5.

$$a = 12$$

$$b = 24$$

13 a $v = 1 - \cos 2x + i\sin 2x$

$$|v| = \sqrt{(1-\cos 2x)^2 + (\sin 2x)^2}$$

$$|v| = \sqrt{1 - 2\cos 2x + \cos^2 2x + \sin^2 2x}$$

$$|v| = \sqrt{1 - 2\cos 2x + 1}$$

$$|v| = \sqrt{2 - 2\cos 2x}$$

$$|v| = \sqrt{2(1 - \cos 2x)}$$

$$|v| = \sqrt{2(2\sin^2 x)}$$

$$|v| = \sqrt{4\sin^2 x}$$

$$|v| = 2\sin x$$

Let $\arg(v) = \alpha$

$$\tan\ \alpha = \frac{\sin 2x}{1 - \cos 2x}$$

$$\tan\alpha = \frac{-2\sin x\cos x}{2\sin^2 x}$$

$$\tan\alpha = -\frac{\cos x}{\sin x}$$

$$\tan\alpha = -\cot x$$

$$\arg(v) = \alpha = -\arctan\left(\tan\left(\frac{\pi}{2} - x\right)\right)$$

$$\arg(v) = x - \frac{\pi}{2}$$

b $$v^{\frac{1}{3}} = (1 - \cos 2x + i\sin 2x)^{\frac{1}{3}} = \left(2\sin x\left(\cos\left(x - \frac{\pi}{2}\right) + i\sin\left(x - \frac{\pi}{2}\right)\right)\right)^{\frac{1}{3}}$$

$$v^{\frac{1}{3}} = 2^{\frac{1}{3}}(\sin x)^{\frac{1}{3}}\left[\cos\left(\frac{x - \frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{x - \frac{\pi}{2} + 2k\pi}{3}\right)\right]$$, where $k = -1, 0, 1$.

11.1 Axiomatic probability systems

1 A bag contains 6 red balls and 4 white balls. Two balls are picked out at random. Find the probability that:

a both were white

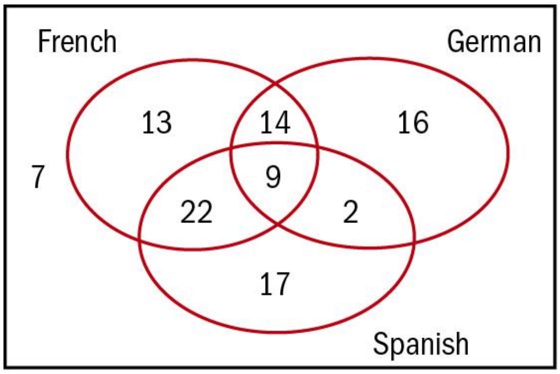
b one was red and the other was white.

2 The events A and B are such that $P(A) = 0.3, P(B) = 0.6$ and $P(A \cap B) = 0.25.$ Find:

a $P(A \cup B)$

b $P(A' \cap B).$

3 This Venn diagram shows the languages studied by 100 students in a year group in a school.



Find the probability that a student chosen at random:

a studies French

b studies French and German

c studies Spanish but no other language

d studies French given that they do not study German.

4 The events A and B are such that $P(A) = 0.4, P(B / A) = 0.7$ and $P(B / A') = 0.5$.

Find:

a $P(B)$

b $P(A \cap B)$.

5 There are three districts in a city. 40% of citizens live in district A, 35% live in district B and 25% live in district C. The unemployment rates of citizens in districts A, B and C are respectively 10%, 8% and 5%.

a Find the probability that a randomly selected citizen is unemployed.

b Given that a randomly selected citizen is unemployed, find the probability that they come from district A.

6 A bag contains 6 blue counters and 4 red counters. I take two counters out at random. My friend John then selects a counter at random. If he selects a blue counter, calculate the probability that both counters I took were red.

Answers

1 **a** $\frac{4}{10}\times\frac{3}{9}=\frac{2}{15}$

b $2\times\frac{4}{10}\times\frac{6}{9}=\frac{8}{15}$

2 **a** 0.65

b 0.35

3 **a** 0.58

b 0.23

c 0.17

d $\frac{F\cap G'}{G'}=\frac{35}{59}$

4 **a** $P(B)=P(B\cap A)+P(B\cap A')$

$$\begin{aligned}&=P(B/A)P(A)+P(B/A')P(A')\\&=0.7\times0.4+0.5\times0.6\\&=0.58\end{aligned}$$

b 0.28

5 **a** $P(U)=P(U/A)P(A)+P(U/B)P(B)+P(U/C)P(C)$

$$\begin{aligned}&=0.1\times0.4+0.08\times0.35+0.05\times0.25\\&=0.0805\end{aligned}$$

b $\frac{P(A\cap U)}{P(U)}=0.4968\ldots$

≈ 0.497

6 By Bayes' Theorem

$$P(RR/B)=\frac{P(B/RR)P(RR)}{P(B/RR)P(RR)+P(B/BB)P(BB)+P(B/RB)P(RB)}$$

$$=\frac{\frac{3}{4}\times\frac{4}{10}\times\frac{3}{9}}{\frac{3}{4}\times\frac{4}{10}\times\frac{3}{9}+\frac{1}{2}\times\frac{6}{10}\times\frac{5}{9}+\frac{5}{8}\times2\times\frac{6}{10}\times\frac{4}{9}}$$

$$= \frac{1}{6}$$

11.2 Probability distributions

1 The discrete random variable X has probability distribution given by

$$P(X = x) = kx(6 - x), x = 1, 2, 3, 4, 5.$$

a Find the value of k.

b Find the mode.

2 Find E(X) for the following probability distribution

x	0	1	2	3	4
P(x=X)	0.05	0.1	0.3	0.4	0.15

3 Two counters are selected at random from a bag containing 5 red counters and 3 blue counters. Find the expected number of red counters selected.

4 A discrete random variable has a PDF given by the following table:

x	1	2	3	4
P(X=x)	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{4}$

Find:

a $\mathrm{E}(X)$

b $\mathrm{E}(2X - 1)$

c $\mathrm{E}(X^2)$

d $\mathrm{Var}(X)$

e the standard deviation of *X*.

5 Two fair dice are rolled and the absolute difference D between the two numbers is noted.

a Construct a probability distribution for D.

b Find E(D) and Var(D).

6 A random variable S has probability distribution given by

$$S = ks^2(5 - s) \qquad s = 2, 3, 4.$$

a Find the value of *k*.

b Find the mode of *S*.

c Find E(*S*) and Var(*S*).

Answers

1 **a** $5k+8k+9k+8k+5k=1$

$\Rightarrow k=\dfrac{1}{35}$

b Mode is 3.

2 0+0.1+0.6+1.2+0.6=2.5

3 Probability distribution is

r	0	1	2
P(R=r)	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

So E(R) = $\dfrac{35}{28}=\dfrac{5}{4}$

4 **a** $\dfrac{1}{12}+\dfrac{1}{3}+\dfrac{3}{2}+1=\dfrac{35}{12}$

b $2\times\dfrac{35}{12}-1=\dfrac{29}{6}$

c $\dfrac{1}{12}+\dfrac{2}{3}+\dfrac{9}{2}+4=\dfrac{37}{4}$

d $\dfrac{37}{4}-\left(\dfrac{35}{12}\right)^2=\dfrac{107}{144}$

e $\sqrt{\dfrac{107}{144}}=0.8620\ldots\approx 0.862$

5 **a**

d	0	1	2	3	4	5
P(D=d)	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

b $\mathrm{E}(D)=\dfrac{35}{18}, \mathrm{Var}(D)=\dfrac{665}{324}$

6 **a** $12k + 18k + 16k = 1 \Rightarrow k = \dfrac{1}{46}$

b Mode $= 3$

c $\mathrm{E}(S) = \dfrac{71}{23}, \mathrm{Var}(S) = \dfrac{318}{529}$

11.3 Continuous random variables

1 A continuous random variable *X* has PDF $f(x)$ where

$$f(x)=\begin{cases} kx(x-1)^2 & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of the constant *k*.

b Find $P(0.3\le X\le 0.7)$.

2 A continuous random variable *X* has PDF $f(x)$ where

$$f(x)=\begin{cases} ax(3-x) & 0\le x\le 3 \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of *a*.

b Find $\mathrm{E}(X)$

c Find $\mathrm{Var}(X)$.

d Find the mode of *X*.

3 A continuous random variable *X* has PDF given by

$$f(x)=\begin{cases} \frac{1}{\pi}x\sin x & 0\le x\le \pi \\ 0 & \text{otherwise} \end{cases}$$

a Find the exact value of $P\left(\frac{\pi}{4}\le x\le \frac{3\pi}{4}\right)$

b Show that $\mathrm{E}(X)=\frac{\pi^2-4}{\pi}$.

c Use the GDC to find the median and mode.

4 A continuous random variable *X* has PDF given by

$$f(x)=\begin{cases} ax^3+\mathrm{bx} & 0\le x\le 2 \\ 0 & \text{otherwise} \end{cases}$$

Given that the median is 1,

a Find *a* and *b*.

b Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

Answers

1 **a** $\int_0^1 kx(x-1)^2 dx = 1$

$\Rightarrow \int_0^1 k\left(x^3 - 2x^2 + x\right)dx = 1$

$\Rightarrow k\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1 = 1$

$\Rightarrow k = 12$

b $\left[3x^4 - 8x^3 + 6x^2\right]_{0.3}^{0.7}$

$= 0.568$

2 **a** $a\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = 1$

$\Rightarrow a = \frac{2}{9}$

b $\int_0^3 \frac{2}{9}x^2(3-x)\mathrm{d}x$

$= \left[\frac{2}{9}x^3 - \frac{1}{18}x^4\right]_0^3$

$= \frac{3}{2}$

c $\mathrm{Var}(X) = \int_0^3 \frac{2}{9}x^3(3-x)\mathrm{d}x - \frac{9}{4}$

$= \left[\frac{1}{6}x^4 - \frac{2}{45}x^4\right]_0^3 - \frac{9}{4}$

$= \frac{9}{20}$

d $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{9}x(3-x)\right) = \frac{2}{3} - \frac{4}{9}x$

$=0$ when $x = \frac{3}{2}$

and this is a maximum because $\frac{d^2y}{dx^2} < 0$

so mode $= \frac{3}{2}$

3 a $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{\pi} x \sin x dx$

Use integration by parts

$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$
$$= \frac{1}{\pi}\left[\frac{3\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$
$$= \frac{1}{\sqrt{2}}$$

b $\int_0^1 \frac{1}{\pi} x^2 \sin x dx$

Use integration by parts twice

$$= \frac{1}{\pi}\left[-x^2\cos x + \int 2x\cos x dx\right]_0^1$$
$$= \frac{1}{\pi}\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^1$$
$$= \frac{1}{\pi}\left[\pi^2 - 4\right]$$
$$= \frac{\pi^2 - 4}{\pi}$$

c $\frac{1}{\pi}\left(-m\cos m + \sin m\right) = 0.5$

$\Rightarrow$ median $= 1.905... \approx 1.91$

$\frac{d}{dx}\left(x\sin x\right) = x\cos x + \sin x$

Mode is when this = 0

By GDC

Mode = 2.028... ≈ 2.03

4 a $\int_0^2 (ax^3 + bx)\,dx = 1$

$\Rightarrow \left[\frac{1}{4}ax^4 + \frac{1}{2}bx^2\right]_0^2 = 1$

$\Rightarrow 4a + 2b = 1$

$\int_0^1 (ax^3 + bx)\,dx = 0.5$

$\Rightarrow \left[\frac{1}{4}ax^4 + \frac{1}{2}bx^2\right]_0^1 = 0.5$

$\Rightarrow a + 2b = 2$

$\Rightarrow a = -\frac{1}{3}, b = \frac{7}{6}$

b $E(X) = \int_0^2 \left(-\frac{1}{3}x^4 + \frac{7}{6}x^2\right)dx$

$= \left[-\frac{1}{15}x^5 + \frac{7}{18}x^3\right]_0^2$

$= \frac{44}{45}$

$\mathrm{Var}(X) = \int_0^2 \left(-\frac{1}{3}x^5 + \frac{7}{6}x^3\right)dx - \left(\frac{44}{45}\right)^2$

$= \left[-\frac{1}{18}x^6 + \frac{7}{24}x^4\right]_0^2 - \left(\frac{44}{45}\right)^2$

$= \frac{314}{2025}$

11.4 Binomial distribution

1 Give that $X \sim B(10, 0.4)$, find

a $P(X = 4)$

b $P(X > 8)$

c $P(3 \leq X \leq 6)$

2 Given that $X \sim B(7, 0.3)$, find:

a $\mathrm{E}(X)$

b $\mathrm{Var}(X)$

c The most likely value of *X*.

3 Given that $X \sim B(6, \frac{1}{2})$:

a Show that $\mathrm{P}(X = 5) = \frac{3}{32}$.

b Construct a probability distribution.

c Find $\mathrm{E}(X)$.

d Find $\mathrm{Var}(X)$.

e Find the mode.

4 The probability that it rains in April on any given day in a UK city is $\frac{2}{3}$. In any given week in April find the probability that it will rain on:

a exactly four days **b** less than three days **c** at least five days.

5 Use your GDC statistical functions to answer this question.

If I roll a fair die 100 times:

a What is the probability that I obtain 10 sixes?

b What is the probability that I obtain more than 20 sixes?

c What is the most likely number of sixes?

Answers

1 a $\binom{10}{4}0.4^4 0.6^6 = 0.2508\ldots \approx 0.251$

b $\binom{10}{9}0.4^9 \times 0.6 + 0.4^{10} = 0.001677\ldots \approx 0.00168$

c $\binom{10}{3}0.4^3 0.6^7 + \binom{10}{4}0.4^4 0.6^6 + \binom{10}{5}0.4^5 0.6^5 + \binom{10}{6}0.4^6 0.6^4$

$= 0.7779\ldots \approx 0.778$

2 a $7 \times 0.3 = 2.1$

b $7 \times 0.3 \times 0.7 = 1.47$

c By inspection P(1)=0.247..., P(2)=0.318..., P(3)=0.226... so most likely value is 2.

3 a

$$\binom{6}{5} \times \left(\frac{1}{2}\right)^6 = \frac{3}{32}$$

b

x	0	1	2	3	4	5	6
P(X=x)	$\frac{1}{64}$	$\frac{3}{32}$	$\frac{15}{64}$	$\frac{5}{16}$	$\frac{15}{64}$	$\frac{3}{32}$	$\frac{1}{64}$

c $6 \times \frac{1}{2} = 3$

d $6 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{2}$

e Mode =3.

4 a $\binom{7}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^3 = \frac{560}{2187}$

b $\left(\frac{1}{3}\right)^7 + 7 \times \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right)^6 + 21 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^5 = \frac{11}{243}$

c $\left(\frac{2}{3}\right)^7 + 7 \times \left(\frac{1}{3}\right) \times \left(\frac{2}{3}\right)^6 + 21 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^5 = \frac{416}{729}$

5 Using the fact that $X \sim B\left(100, \frac{1}{6}\right)$ and GDC

a $0.002140... \approx 0.00214$

b $0.1518... \approx 0.152$

c By inspection, $P(15) = 0.1002..., P(16) = 0.1065...$

and $P(16) = 0.1052...$

so most likely value is 16.

11.5 The normal distribution

1 Given $X \sim N(3, 5^2)$, find:

a $P(1.6 \le X \le 2.6)$

b $P(X > 4)$

c $P(X \le 3.2)$

2 Bags of sugar have weights normally distributed with mean 1010g and standard deviation 4g. If a bag is selected at random, what is the probability that it weighs less than 1kg?

3 Given $X \sim N(17, 3.6)$:

a Find $P(14 \le X \le 18)$

b Find *x* if $P(X < x) = 0.07$

4 If $X \sim N(10, \sigma^2)$, find the value of σ^2 given that $P(X \ge 11) = 0.372$.

5 If $X \sim N(\mu, \sigma^2)$, find the values of μ and σ, given that $P(X \le 3) = 0.271$ and $P(X \ge 8) = 0.954$.

6 Bags of cement are labelled as weighing 50kg. If the mean weight of the bags is 50.5kg, what value of σ^2 will ensure that 95% of the bags weigh at least 50kg?

Answers

1 **a** $0.003100... \approx 0.00310$

b $0.4207... \approx 0.421$

c $0.5159... \approx 0.516$

2 $0.006209... \approx 0.00621$

3 **a** $0.6439... \approx 0.644$

b $14.19... \approx 14.2$

4

$$\frac{1}{\sigma} = 0.3265...$$
$$\Rightarrow \sigma^2 = 9.377... \approx 9.38$$

5

$$\frac{3-\mu}{\sigma} = -0.6097... \Rightarrow 3-\mu = -0.6097...\sigma$$
$$\frac{8-\mu}{\sigma} = 1.6849... \Rightarrow 8-\mu = 1.6849...\sigma$$
$$\Rightarrow \sigma = 2.178... \approx 2.18$$
$$\mu = 4.328... \approx 4.33$$

6 $\frac{0.5}{\sigma} = 1.644... \Rightarrow \sigma^2 = 0.09240 \approx 0.0924$